JOURNAL OF FUNCTIONAL ANALYSIS **83**, 1–43 (1989)

Infinite Complex Spin Groups

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Communicated by L. Gross

Received September 1, 1986

INTRODUCTION

Let W denote a complex even dimensional vector space with a distinguished non-degenerate bilinear form $(\cdot,\cdot)$. The complex orthogonal group $O_{\mathbb{C}}(W)$ is the group of linear transformations on W which preserve the bilinear form $(\cdot,\cdot)$. The subgroup of $O_{\mathbb{C}}(W)$ with determinant 1 is the connected component of the identity $SO_{\mathbb{C}}(W)$. It is well known [3] that $SO_{\mathbb{C}}(W)$ has a simply connected twofold covering group $\mathrm{Spin}_{\mathbb{C}}(W)$. There is a homomorphism $T\colon \mathrm{Spin}_{\mathbb{C}}(W) \to SO_{\mathbb{C}}(W)$ and an exact sequence

$$\mathbb{Z}/2\mathbb{Z} \longrightarrow \mathrm{Spin}_{\mathbb{C}}(W) \xrightarrow{\;T\;} SO_{\mathbb{C}}(W) \longrightarrow 0. \tag{0.1}$$

The group $\mathrm{Spin}_{\mathbb{C}}(W)$ may be realized explicitly as a subgroup of the group of automorphisms of the Clifford algebra $\mathscr{C}(W)$. Here $\mathscr{C}(W)$ is the associative algebra with unit e generated by the elements of W subject to the relations $xy + yx = (x, y)e$ where $x, y \in W$. The automorphism group of $\mathscr{C}(W)$ contains a subgroup which leaves W invariant. It is in this subgroup that one finds a model for $\mathrm{Spin}_{\mathbb{C}}(W)$ (these well-known facts are reviewed in detail in Section 2).

In this paper we are interested in extensions of this construction to infinite dimensional spaces W. For technical reasons we consider Hilbert spaces W in which the distinguished bilinear form $(\cdot,\cdot) = \langle \cdot, P\cdot \rangle$ is derived

* Research supported in part by NSF Grant DMS–8421289.

0022-1236/89 \$3.00

from the Hermitian inner product $\langle \cdot,\cdot \rangle$ through a conjugation P on W. In the infinite dimensional case there is a C^*-algebra $\mathscr{C}(W)$ [36] in which one can locate an infinite dimensional analogue of the finite dimensional spin group. However, the group that one obtains in this fashion is rather small; too small in fact to give a complete picture of the situation inside the Fock representations which are of principal interest for us. We now describe these representations. A subspace V of W will be said to be isotropic if the bilinear form $(\cdot,\cdot)$ vanishes identically on V. Associated with every orthogonal direct sum decomposition of $W = W_+ + W_-$ into isotropic subspaces $W_\pm$ there is a distinguished representation of the Clifford algebra $\mathscr{C}(W)$. We let Q denote the map on W which is multiplication by $+1$ on W_+ and multiplication by -1 on W_-. We write $\mathscr{C}(W) \ni X \to F_Q(X)$ for the associated representation of $\mathscr{C}(W)$ and refer to F_Q as the Q-Fock representation of $\mathscr{C}(W)$. For $w \in W \subseteq \mathscr{C}(W)$ the operator $F_Q(w)$ is the sum of creation and annihilation operators acting on the alternating tensor algebra $A(W_+)$ over W_+ (this is also considered in more detail in Section 2). For finite dimensional spaces W all Fock representations are equivalent as are the associated representations of $\mathrm{Spin}_{\mathbb{C}}(W)$. When W is infinite dimensional the Fock representations associated with Q_1 and Q_2 are equivalent if and only if $Q_2 - Q_1$ is a Schmidt class operator [12]. Turning to the associated representation of the spin group it is natural to try to characterize the complex orthogonals G on W for which there exists an invertible map g on $A(W_+)$ with $gF_Q(w)\, g^{-1} = F_Q(Gw)$, $w \in W$. Because the representation class of F_Q varies with Q one may expect that this family of complex orthogonals also depends on Q. When G is a real orthogonal (i.e., one that commutes with P) then it is well known that a unitary g on $A(W_+)$ exists with $gF_Q(w)\, g^{-1} = F_Q(Gw)$ if and only if the commutator $GQ - QG$ is in the Schmidt class [36]. One of our principal motivations in this paper is an extension of this result to the full *complex* orthogonal group. In making such an extension to the complex orthogonal group one must give up the requirement that the maps g are everywhere defined on $A(W_+)$. Instead we find that there is a *common* dense invariant domain $\mathscr{D}$ contained in $A(W_+)$ on which all the transformations g act. Most of the technical complications in this paper arise from the fact that the operators g are unbounded on $A(W_+)$ and one must construct the domain $\mathscr{D}$. The Hilbert space structure on $A(W_+)$ plays an auxiliary role in this paper and it is conceptually better to think of the representation we exhibit in terms of the Borel–Weil construction for representations of compact Lie groups. The irreducible representations of a compact semi-simple Lie group K may be realized in terms of the action of K on the sections of a holomorphic line bundle over a homogeneous space for K [38]. In this paper one should think of the real compact Lie group $K = \mathrm{Spin}_{\mathbb{R}}(W)$. In some sense what survives the transition to infinite dimensional W and the complexification of K

to $\mathrm{Spin}_{\mathbb{C}}(W)$ is the action of $\mathrm{Spin}_{\mathbb{C}}(W)$ on a holomorphic line bundle over a homogeneous space for $\mathrm{Spin}_{\mathbb{R}}(W)$. This is explained further in Section 5. Indeed the principal motivation for our work on this problem arose from the construction of a related representation in a paper by G. Segal and G. Wilson [35]. As is explained in Section 4 the group which is considered in [35] may be regarded as a subgroup of the group we construct here. When "restricted" to this subgroup the holomorphic line bundle of interest has a nice geometric description as the "renormalized" det* bundle over a Grassmannian. Segal and Wilson construct their group by first constructing the det* bundle. This is a more direct and illuminating construction than the one we present here. There are two reasons we chose not to imitate their procedure. The first is that we did not know how to "geometrize" the description of the "Pfaffian" bundle which arises for the full spin group as effectively as Segal and Wilson did for the det* bundle. The second reason is that we wished to exhibit the connection with Clifford algebras. Some work is required to bring Clifford algebras into the picture if one starts with Segal and Wilson's construction.

We now describe in more detail the structure of the group which we construct. Most of the work in Section 3 is devoted to the construction of a topological group $\mathrm{Spin}_Q(W)$ as a direct limit of finite dimensional complex spin groups in which the exact sequence (0.1) survives:

$$\mathbb{Z}/2\mathbb{Z} \longrightarrow \mathrm{Spin}_Q(W) \xrightarrow{T} SO_Q(W) \longrightarrow 0.$$

Here $SO_Q(W)$ is the connected component of the identity in the group of complex orthogonals G on W with matrices $G = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ relative to the $W_+ \oplus W_-$ decomposition of W such that $\left[\begin{smallmatrix} A & O \\ O & D \end{smallmatrix}\right]$ is a trace class perturbation of the identity and $\left[\begin{smallmatrix} O & B \\ C & O \end{smallmatrix}\right]$ is a Schmidt class operator. We also construct a dense linear subspace $\mathscr{D} \subseteq A(W_+)$ and a group homomorphism $\Gamma_Q : \mathrm{Spin}_Q(W) \to L(\mathscr{D})$ (where $L(\mathscr{D})$ is the space of linear maps on $\mathscr{D}$ which leave $\mathscr{D}$ invariant) such that

$$\Gamma_Q(g)\, F_Q(w)\, \Gamma_Q(g)^{-1} = F_Q(T(g)\, w), \qquad w \in W. \tag{0.2}$$

The group of complex orthogonals which commute with Q is also important. It is easy to see that this group is isomorphic to the general linear group $GL(W_+)$. It is straightforward to construct a homomorphism $\Gamma\colon GL(W_+) \to L(\mathscr{D})$ such that

$$\Gamma(G)\, F_Q(w)\, \Gamma(G)^{-1} = F_Q(G \oplus G^{-\tau} w). \tag{0.3}$$

We also show that the elements G of $GL(W_+)$ act on $\mathrm{Spin}_Q(W)$ via an automorphism $\alpha(G)$ and we have

$$\Gamma(G)\, \Gamma_Q(g) \Gamma(G)^{-1} = \Gamma_Q(\alpha(G)\, g). \tag{0.4}$$

Thus $\Gamma_Q \times \Gamma$ gives a representation of the semi-direct product $\mathrm{Spin}_Q(W) \times_\alpha GL(W_+)$ on $L(\mathscr{D})$. Let ker denote the kernel of the homomorphism $\Gamma_Q \times \Gamma$ and write $\widehat{\mathrm{Spin}}_Q(W) = \mathrm{Spin}_Q(W) \times_\alpha GL(W_+)/\mathrm{ker}$ for the quotient group. The map $T(g \times G) = T(g) \cdot G \oplus G^{-\tau}$ induces a homomorphism on the quotient $\widehat{\mathrm{Spin}}_Q(W)$ for which one has the exact sequence

$$\mathbb{C}^* \longrightarrow \widehat{\mathrm{Spin}}_Q(W) \xrightarrow{T} SO_{\mathrm{res}}(W) \longrightarrow 0, \tag{0.5}$$

where $SO_{\mathrm{res}}(W)$, the restricted special orthogonals, is the connected component of the identity in the group of complex orthogonals $O_{\mathrm{res}}(W)$ which have Schmidt class commutators with Q. The exact sequence (0.5) reveals the principal difference between the finite dimensional situation and the infinite dimensional one. The homomorphism T is no longer a twofold covering but an extension of $SO_{\mathrm{res}}(W)$ by $\mathbb{C}^*$, the multiplicative group of non-zero complex numbers. The two homomorphisms Γ_Q and Γ "interact" through a non-trivial cocycle.

Let $\hat{\Gamma}_Q$ denote the homomorphism from $\widehat{\mathrm{Spin}}_Q(W)$ into $L(\mathscr{D})$ induced by $\Gamma_Q \times \Gamma$. Then as an identity on $\mathscr{D}$ one has a consequence of (0.2) and (0.3):

$$\hat{\Gamma}_Q(g)\, F_Q(W)\, \hat{\Gamma}_Q(g)^{-1} = F_Q(T(g)\, W), \qquad w \in W,\ g \in \widehat{\mathrm{Spin}}_Q(W). \tag{0.6}$$

The construction of the group $\widehat{\mathrm{Spin}}_Q(W)$ and the representation $\hat{\Gamma}_Q$ satisfying (0.6) is the principal result of this paper.

The action of the Clifford algebra allows us to extend these considerations to $O_{\mathrm{res}}(W)$. Every element of $O_{\mathrm{res}}(W)$ differs from an element of $SO_{\mathrm{res}}(W)$ by the action of a single complex orthogonal reflection on W. The elements of the Clifford algebra $w \in W \subseteq \mathscr{C}(W)$ with $(w, w) \neq 0$ implement such complex orthogonal reflections in the Fock representation (except for a sign change which is discussed in Section 3). Thus by combining the action of the Clifford algebra and the representation of $\widehat{\mathrm{Spin}}_Q(W)$ we are able to cover the action of $O_{\mathrm{res}}(W)$ in the Fock representation.

Because we concentrate on the complex group $\widehat{\mathrm{Spin}}_Q(W)$ we do not consider the unitary structure associated with the real subgroup $SO_{\mathbb{R},Q}(W)$ of $SO_Q(W)$. We would like to mention, however, that in a series of papers [24, 25, 26] D. Pickrell has shown how this unitary structure arises from a measure "on" the appropriate homogeneous space. This is important for a more complete understanding of these representations from the Borel–Weil point of view.

Our interest in these matters was stimulated largely by the connection these groups have with a class of two dimensional "exactly solvable" models in quantum field theory and statistical mechanics. Among these models are the Federbush model [31], the massless Thirring model [32], the Luttinger model [4], the XY spin chain [34], and the two dimensional Ising model [15, 16, 17]. In each case the field operators are either in $\widehat{\mathrm{Spin}}_Q(W)$ or are singular limits of elements in $\widehat{\mathrm{Spin}}_Q(W)$. The Fock representation of $\widehat{\mathrm{Spin}}_Q(W)$ also contains many of the more interesting representations of Kac–Moody algebras and current algebras [6]. The paper of Segal and Wilson [35] mentioned earlier has an attractive presentation of the relations with KdV type hierarchies discovered by M. Sato and Y. Sato and further developed by Date, Jimbo, Kashiwara, and Miwa [8]. The τ-functions for monodromy preserving deformations introduced by Sato, Miwa, and Jimbo [8; referred to as S.M.J. in this paper] in their monumental study of the scaling limit of the two dimensional Ising model are vacuum expectations of (singular) elements from $\widehat{\mathrm{Spin}}_Q(W)$. The improved understanding of the group structure for $\widehat{\mathrm{Spin}}_Q(W)$ has already made possible a lattice approach to the S.M.J. analysis in [18] which is a generalization (and in some respects a simplification) of earlier work [17] rigorously justifying the scaling limit analysis of the Ising model. Malgrange [11] has clarified the existence of the τ-function for monodromy preserving deformations of the Cauchy–Riemann equations. This work depends on a somewhat mysterious formula for $d(\log \tau)$ discovered by S.M.J. [8]. One of our ambitions is a lattice route to such a formula which we believe ought to clarify the representational significance of the τ-function. Related to this is the application of generalized Wick theorems to non-linear difference identities for Ising correlations (the McCoy, Wu, and Perk difference identities [14]) and to generalizations to lattice monodromy fields [19]. We also expect to present a proof of the scaling hypothesis for monodromy fields in the near future (as explained in [20]) based in part on the improvements in the description of $\widehat{\mathrm{Spin}}_Q(W)$ presented here.

To conclude this introduction we would like to acknowledge useful conversations with R. Richardson and D. Pickrell. In particular D. Pickrell's thesis [25] contains many of the ideas for the real spin group that are worked out here for the complex spin group.

1. The Complex Restricted Orthogonal Group

In this section we assemble some facts about $SO_Q(W)$ and $O_{\mathrm{res}}(W)$ that will be of use to us in subsequent developments. Let W denote a complex Hilbert space with Hermitian inner product $\langle \cdot,\cdot \rangle$ and distinguished conjugation P. The space W then has a non-degenerate bilinear form $(\cdot,\cdot) = \langle \cdot, P \cdot \rangle$. Let Q denote a symmetric (with respect to $\langle \cdot,\cdot \rangle$) involution on W which anti-commutes with P. Let $Q_\pm = (1 + Q)/2$ and $W_\pm = Q_\pm W$. Then $W = W_+ \oplus W_-$ is an orthogonal direct sum decomposition of W into isotropic subspaces for $(\cdot,\cdot)$. If G is an operator on W then we write $G=\left[\begin{smallmatrix} A(G) & B(G) \\ C(G) & D(G)\end{smallmatrix}\right]$ for the matrix of G relative to the $W_+ \oplus W_-$ decomposition of W (here $A(G)\colon W_+ \to W_+$, $B(G)\colon W_- \to W_+$, etc.). A linear map G on W is orthogonal if it preserves the bilinear form $(\cdot,\cdot)$. This is the same as $G^\tau G = I$ where G^τ is the transpose of G relative to $(\cdot,\cdot)$. The condition that $GQ-QG$ is a Schmidt class operator is the same as the condition that $\left[\begin{smallmatrix} 0 & B(G) \\ C(G) & 0\end{smallmatrix}\right]$ is in the Schmidt class. This implies that $\left[\begin{smallmatrix} A(G) & 0 \\ 0 & D(G)\end{smallmatrix}\right] = G - \left[\begin{smallmatrix} 0 & B(G) \\ C(G) & 0\end{smallmatrix}\right]$ is a Fredholm operator with index 0 since it is a compact perturbation of an invertible operator. Thus $A(G)$ and $D(G)$ are Fredholm and $\mathrm{ind}[A(G)] + \mathrm{ind}[D(G)] = 0$. Since G is complex orthogonal $D(G)^\tau A(G) = I - B(G)^\tau C(G)$ is a compact perturbation of the identity on W_+ and hence has index 0. But $\mathrm{ind}(D^\tau) = -\mathrm{ind}(D)$ so that $\mathrm{ind}[A(G)] - \mathrm{ind}[D(G)] = 0$. Thus if G is orthogonal and $GQ-QG$ is a Schmidt class operator, it follows that $\mathrm{ind}[A(G)] = \mathrm{ind}[D(G)] = 0$. We now define

$$O_{\mathrm{res}}(W) = \{G \mid G^\tau G = I \text{ and } QG - GQ \in \text{Schmidt class}\}.$$

A topology on $O_{\mathrm{res}}(W)$ is determined by the metric

$$\rho(G_1, G_2) = \|G_1 - G_2\| + \|[Q, G_1 - G_2]\|_2,$$

where $\|\cdot\|$ is the usual operator norm and $\|\cdot\|_2$ is the Schmidt norm. For $G \in O_{\mathrm{res}}(W)$ we write $G = U|G|$ for the polar decomposition and note that U commutes with P. The proof of Lemma 4.3 of [5] applies here to show that the real orthogonals in $O_{\mathrm{res}}(W)$ (i.e., those commuting with P) form a retract of $O_{\mathrm{res}}(W)$. In particular G and U are connected by a path in $O_{\mathrm{res}}(W)$. The group of real orthogonals in $O_{\mathrm{res}}(W)$ has just two connected components labelled by the homomorphism i onto $\mathbb{Z}/2\mathbb{Z}$ given by

$$i(U) = \dim \ker D(U) \qquad (\mathrm{mod}\ 2).$$

Hence the homomorphism $G \to \dim \ker D(G)$ (mod 2) labels the connected components of $O_{\mathrm{res}}(W)$. In fact more is true, namely that $O_{\mathrm{res}}(W)$ has the homotopy type of the homogeneous space $O(\infty)/U(\infty)$ where $O(\infty)$ and $U(\infty)$ are the stable orthogonal and unitary groups, respectively (cf. Theorem 1.4 of [5]). A corollary is that $\pi_1(O_{\mathrm{res}}(W))$ is $\mathbb{Z}/2\mathbb{Z}$.

We define $SO_{\mathrm{res}}(W)$ as the kernel of the homomorphism $G \to \dim\ker D(G)$ (mod 2). Thus $SO_{\mathrm{res}}(W)$ consists of those $G \in O_{\mathrm{res}}(W)$ with $\ker(D(G))$ of even dimension. Of special interest for us is the subgroup $SO_Q(W)$ consisting of those $G \in SO_{\mathrm{res}}(W)$ with diagonal part $A(G) \oplus D(G) = (Id + \text{trace class})$ on W. Since every Fredholm operator of index 0 is invertible modulo the trace class we can for each $G \in SO_{\mathrm{res}}(W)$ find an invertible $F\colon W_+ \to W_+$ so that $A(G)F^{-1} = I + \text{trace class}$. The map $F^{-1} \oplus F^\tau$ is a complex orthogonal which commutes with Q, and we define

$G' = G(F^{-1} \oplus F^{\tau})$. Then G' is in $SO_{\mathrm{res}}(W)$ and $A(G') = I +$ trace class. However, since G' is a complex orthogonal it follows that $D(G')^{\tau} A(G') + B(G')^{\tau} C(G') = I$. Since $B^{\tau}C$ is a trace class operator it follows that $D(G')^{\tau}$ and hence also $D(G')$ is of the form $I +$ trace class. Thus $G' \in SO_Q(W)$. We have shown that $SO_{\mathrm{res}}(W)$ is the product of $SO_Q(W)$ and the contractible subgroup consisting of complex orthogonals commuting with Q.

A result that will be of use to us in characterizing an orbit space which arises later in the paper is:

LEMMA 1.1. *There is a decomposition*

$$SO_Q(W) = SO_{\mathbb{R},Q}(W)\, B_Q,$$

where B_Q is the subgroup of $SO_Q(W)$ consisting of lower triangular operators relative to the decomposition $W_+ \oplus W_-$ and $SO_{\mathbb{R},Q}(W)$ is the subgroup consisting of real orthogonal operators. This decomposition of G into a product UL with $U \in SO_{\mathbb{R},Q}(W)$ and $L \in B_Q$ is continuous and unique if we choose L so that $D(L) \geqslant 0$.

Remark. The subset of B_Q satisfying $D(L) \geqslant 0$ is contractible so this provides a direct proof that $SO_Q(W)$ has the homotopy type of $SO_{\mathbb{R},Q}(W)$.

Proof of Lemma. Let $G^*G = \left[\begin{smallmatrix} A & B \\ B^* & D \end{smallmatrix}\right]$ relative to the decomposition $W = W_+ \oplus W_-$, where $G \in SO_Q(W)$. The operators A and D are non-negative self adjoint operators. In fact, since they are both of the form $I +$ trace class, they must be invertible. If D was not invertible for example then it would have 0 as an eigenvalue. The quadratic form associated with G^*G would then vanish on the corresponding eigenvector implying that 0 was in the spectrum of G^*G. This is not possible since G is invertible. Let $\alpha = D^{1/2}$ be the positive square root of D and write $\beta = B\alpha^{-1}$. Let

$$L^{-1} = \begin{bmatrix} \bar{\alpha} & 0 \\ \beta & \alpha^{-1} \end{bmatrix},$$

where we write $\bar{X} = PXP$. Since G^*G is complex orthogonal it follows that $\bar{B}\bar{D}^{-1} + D^{-1}B^{\tau} = 0$ and from this it follows that L^{-1} (and hence L) is complex orthogonal. But

$$G^*GL^{-1} = \begin{bmatrix} \bar{\alpha}^{-1} & \beta \\ 0 & \alpha \end{bmatrix} = \begin{bmatrix} \bar{\alpha} & 0 \\ \beta & \alpha^{-1} \end{bmatrix}^{*-1}$$

or $GL^{-1} = G^{*-1}L^*$. Thus $(GL^{-1})^* = (GL^{-1})^{-1}$ so that $GL^{-1} = U$ is real orthogonal. Uniqueness is straightforward and to prove continuity it is enough to check that the map

$$G \to D(G^*G)^{1/2}$$

is continuous. Now $D(G^*G)>0$ so that if $G_n \to G$ in $SO_Q(W)$ we have $D(G_n{}^*G_n) \to D(G^*G)$ in trace norm and we may thus suppose $D(G_n{}^*G_n)$ is bounded uniformly away from zero for all n. A resolvent integral representation for the square root then shows that $D(G_n{}^*G_n)^{1/2}$ converges to $D(G^*G)^{1/2}$ in trace norm. Q.E.D.

A technical result that we make frequent use of in Section 3 is:

LEMMA 1.2. *Suppose $G \in SO_Q(W)$. Then there is an operator $B \in SO_Q(W)$ with matrix $\left(\begin{smallmatrix} I & 0 \\ b & I \end{smallmatrix}\right)$ relative to the $W_+ \oplus W_-$ decomposition of W with b finite rank and of arbitrarily small norm such that $D(BGB^{-1})$ is invertible.*

Proof. Let $G = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$. We must choose b so that $bB + D$ is non-singular. From the relation $D^\tau B + B^\tau D = 0$ we see that B maps $\ker D$ into $\ker D^\tau$. From $C^\tau B + A^\tau D = I$ it follows that the restriction of B to $\ker D$ is non-singular. Since $\operatorname{ind}(D) = 0$ it follows that $\dim \ker D = \dim \ker D^* = \dim \ker D^\tau$ so that B defines a vector space isomorphism of $\ker D$ onto $\ker D^\tau$. Thus we choose b to map $\ker D^\tau$ onto $\ker D^* =$ the orthogonal complement of the range of D. Observe however that $b^\tau = -b$ is necessary for $B = \left[\begin{smallmatrix} I & 0 \\ b & I \end{smallmatrix}\right]$ to be orthogonal. A non-singular b satisfying this condition clearly exists only if $\dim \ker D^\tau$ is even. This is true since $G \in SO_{\mathrm{res}}(W)$. One may extend b to be zero on $(\ker D^\tau)^\perp$ without effecting the skew symmetry. Observe moreover that b may be chosen with arbitrarily small norm. Q.E.D.

The following corollary is also useful:

COROLLARY 1.3. *Suppose $G \in SO_Q(W)$, then there is an $H \in SO_Q(W)$ with $H = I +$ finite rank, such that $D(HGH^{-1})$ and $D(HG^{-1}H^{-1})$ are both invertible.*

Proof. Let $G = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$. Since $G^{-1} = \left[\begin{smallmatrix} D^\tau & B^\tau \\ C^\tau & A^\tau \end{smallmatrix}\right]$ it is clearly enough to find H so that $A(HGH^{-1})$ and $D(HGH^{-1})$ are both invertible. By imitating the proof of Lemma 1.2 we can find an *upper triangular* finite rank perturbation of the identity in $SO_Q(W)$ which conjugates G to an element with invertible A matrix element. Now apply Lemma 1.2 to the result of this transformation to make the D matrix element invertible. One might of worry that the A matrix element has again become singular but because the norm of b in Lemma 1.2 can be made arbitrarily small this problem can clearly be avoided. Q.E.D.

2. THE FINITE SPIN GROUP $\mathrm{SPIN}_{\mathbb{C}}\, W$

Let W denote an even dimensional complex vector space with a non-degenerate bilinear form $(\cdot,\cdot)$. The Clifford algebra, $\mathscr{C}(W)$, is the associative algebra with identity that is generated by the identity, e, and the elements of W subject to the multiplicative relations:

$$w_1 w_2 + w_2 w_1 = (w_1, w_2)e, \qquad w_1, w_2 \in W.$$

A subspace V of W is said to be isotropic if $(v_1, v_2) = 0$ for any pair $v_1, v_2 \in V$. Associated with every splitting $W = W_+ \oplus W_-$ as a direct sum of isotropic subspaces $W_\pm$, there is a representation of $\mathscr{C}(W)$ which we now describe. Let $A(W_+)$ denote the complex alternating tensor algebra over W_+. That is $A(W_+) = \mathbb{C} \oplus \sum_k A^k(W_+)$ where $A^k(W_+)$ denotes the space of alternating k tensors over W_+ and the sum ranges from $k = 1$ to $k = \dim W_+$. For each $x \in W_+$ define a map $C(x)$ on $A(W_+)$ by

$$C(x)v = x \wedge v, \qquad x \in W_+, v \in A(W_+).$$

There are a number of definitions of the wedge product. The normalization will not be important for us but for definiteness we adopt the usage in Arnold [1]. Since each of the subspaces $W_\pm$ is isotropic and the bilinear form is assumed to be non-degenerate it follows that the bilinear pairing of W_+ with W_- induces an isomorphism of W_+ with W_-^* (the dual of W_-) and vice versa. Thus we may also identify $A(W_+)$ with the dual space $A(W_-)^*$. There is a certain arbitrariness in this identification but here we will be quite specific. Let $\{e_1, ..., e_N\}$ be a complex basis for W_+ and let $\{e_1^*, ..., e_N^*\}$ be the dual basis for W_- (that is $(e_i, e_j^*) = \delta_{ij}$). The vectors 1 and $e_{i_1} \wedge \cdots \wedge e_{i_k}$, $i_1 < i_2 \cdots < i_k$, are a basis for $A(W_+)$ and the vectors 1 and $e_{i_1}^* \wedge e_{i_2}^* \cdots \wedge e_{i_k}^*$, $i_1 < \cdots < i_k$, are a basis for $A(W_-)$. We choose the pairing between $A(W_+)$ and $A(W_-)$ which identifies $A(W_+)$ with the dual $A(W_-)^*$ so that the bases just described are dual bases. The reader may check that this pairing is natural in the sense that it does not depend on the choice of basis $\{e_i\}_{i=1}^N$ for W_+. For $v \in A(W_-)$ and $y \in W_-$ define $C(y)v = y \wedge v$. Let $C^\tau(y)$ denote the transpose (dual) map from $A(W_-)^*$ to $A(W_-)^*$. Using the identification of $A(W_-)^*$ with $A(W_+)$ described above we may regard $C^\tau(y)$ as a map on $A(W_+)$. With this understood we define

$$F(x) = C(x_+) + C^\tau(x_-), \qquad x = x_+ + x_-, \; x_\pm \in W_\pm.$$

We leave it to the reader to check that $e \to I$ and $x \to F(x)$ extends to a representation $\mathscr{C}(W)$ on $A(W_+)$. This is the Fock representation of $\mathscr{C}(W)$ associated with the splitting $W = W_+ + W_-$. Since we will often consider more than one such splitting it will be convenient to introduce a "parametrization" of such splittings. For $x = x_+ + x_-$ $(x_\pm \in W_\pm)$ define $Q_\pm x = x_\pm$ and let $Q = Q_+ - Q_-$. It is easy to check that Q is a skew symmetric involution. That is $Q^\tau = -Q$ and $Q^2 = I$ where Q^τ is the transpose of

Q relative to the bilinear form $(\cdot,\cdot)$. Conversely if Q is a skew symmetric involution and one defines $Q_\pm = (1 \pm Q)/2$, then $W_+ = Q_+ W$ and $W_- = Q_- W$ give an isotropic splitting of W. We call the representation of $\mathscr{C}(W)$ just constructed the Q-Fock representation of $\mathscr{C}(W)$. To emphasize the dependence on Q we sometimes write

$$F_Q(x) = C(x_+) + C^\tau(x_-).$$

Next we introduce the Clifford group $G(W)$. This consists of the invertible elements $g \in \mathscr{C}(W)$ such that $gxg^{-1} = Gx$ for $x \in W \subseteq \mathscr{C}(W)$ and $G\colon W \to W$ an orthogonal map (i.e., $G^\tau G = I$). Suppose $w \in W$ is such that $(w, w) \neq 0$. Then

$$wxw^{-1} = 2\frac{(x, w)}{(w, w)} w - x.$$

The map $x \to x - 2((x, w)/(w, w))w$ is the orthogonal reflection through the hyperplane orthogonal to w. Let $\{e_k\}_{k=1}^N$ be an orthonormal basis for W (i.e., $(e_i, e_j) = \delta_{ij}$) and choose a constant C so that $\Omega = Ce_1 e_2 \cdots e_N$ has the property $\Omega^2 = 1$. It is easy to check that since dim W is even

$$\Omega x \Omega = -x.$$

The Cartan–Dieudonné theorem [2] informs us that every complex orthogonal is the product of reflections. By taking products $w_1 \cdots w_k$, with perhaps an additional factor of Ω to straighten out the minus sign in an odd product of reflections we see that for every complex orthogonal G, there exists a $g \in G(W)$ such that $gxg^{-1} = Gx$. If $g_j \in G(W)$ $(j = 1, 2)$ and $g_1 x g_1^{-1} = Gx = g_2 x g_2^{-1}$ for $x \in W$ then $g_1^{-1} g_2$ commutes with $W \subseteq \mathscr{C}(W)$ and hence with all of $\mathscr{C}(W)$. For W even dimensional $\mathscr{C}(W)$ is a simple algebra (in fact every Fock representation of $\mathscr{C}(W)$ establishes an isomorphism between $\mathscr{C}(W)$ and the full matrix algebra over $A(W_+)$). Hence $g_1^{-1} g_2 = (\text{const.})e$. Thus every element $g \in G(W)$ can be written as a product $w_1 \cdots w_k$ for some elements $w_j \in W$ such that $(w_j, w_j) \neq 0$. It follows that every element of $G(W)$ is either even or odd in $\mathscr{C}(W)$. Following S.M.J. [8] we define $T(g)$, the induced rotation associated with $g \in G(W)$, as

$$gxg^{-1} = T(g)x \qquad \text{for} \quad x \in W,\ g \text{ even in } \mathscr{C}(W)$$

$$gxg^{-1} = -T(g)x \qquad \text{for} \quad x \in W,\ g \text{ odd in } \mathscr{C}(W).$$

This makes the induced rotation for $w \in W$ $((w, w) \neq 0)$ equal to the reflection in the hyperplane orthogonal to w rather than to minus the reflection and it remains true that $T\colon G(W) \to O_{\mathbb{C}}(W)$ is a homomorphism.

On $\mathscr{C}(W)$ there is a *linear* involution which extends the identity on e and W. We write $g \to g^{\tau}$ to denote this involution. Suppose $g = w_1 \cdots w_k$ is an element of $G(W)$, then $g^{\tau}g = \prod_{j=1}^{k} w_j^2$ is a non-zero scalar and it follows easily that $nr(g) \overset{\text{def}}{=} g^{\tau}g$ is a homomorphism from $G(W)$ to $\mathbb{C}^*$ (this is called the spinor norm in [8]). The kernel of this homomorphism is called $\mathrm{Pin}_{\mathbb{C}}(W)$ and we have the exact sequence

$$\mathbb{Z}/2\mathbb{Z} \longrightarrow \mathrm{Pin}_{\mathbb{C}}(W) \xrightarrow{T} O_{\mathbb{C}}(W) \longrightarrow 0.$$

If we restrict ourselves to the connected component of the identity $SO_{\mathbb{C}}(W)$ in $O_{\mathbb{C}}(W)$ this defines $\mathrm{Spin}_{\mathbb{C}}(W) \subseteq G(W)$ and we have

$$\mathbb{Z}/2\mathbb{Z} \longrightarrow \mathrm{Spin}_{\mathbb{C}}(W) \xrightarrow{T} SO_{\mathbb{C}}(W) \longrightarrow 0.$$

We shall be concerned with constructing infinite dimensional versions of $\mathrm{Spin}_{\mathbb{C}}(W)$ that are well adapted to Fock representations. To do this we need a more detailed picture of $\mathrm{Spin}_{\mathbb{C}}(W)$ as it sits inside the Fock representation. Suppose Q is a skew symmetric involution on W and $W_{\pm} = Q_{\pm} W$ is the associated isotropic splitting. Let $1_Q = 1 \oplus 0 \oplus 0 \cdots \oplus 0$ be the vacuum vector in $A(W_+)$. The Q-Fock state on $\mathscr{C}(W)$ is

$$\langle g \rangle_Q = \langle F_Q(g)\, 1_Q, 1_Q^* \rangle, \qquad g \in \mathscr{C}(W),$$

where $1_Q^* = 1 \oplus 0 \oplus \cdots \oplus 0 \in A(W_-) \simeq A(W_+)^*$ and $\langle \cdot, \cdot \rangle$ denotes the dual pairing described above.

Our first result is a formula for the elements g of $G(W)$ such that $\langle g \rangle_Q \neq 0$. This formula will provide us with local charts for $\mathrm{Spin}_{\mathbb{C}}(W)$ and by varying Q we will be able to cover the whole group. We first prepare some notation. The fact that the elements of $W_{\pm}$ anti-commute with themselves allows one to define a linear map θ_Q from the Grassmann algebra $A(W)$ to the Clifford algebra $\mathscr{C}(W)$ with the following properties:

(1) $\theta_Q 1 = 1$, $\theta_Q(x) = x$, $x \in W$;

(2) $\theta_Q(A \wedge B) = \theta_Q(A) \cdot \theta_Q(B)$ if $A \in A(W_+)$ or $B \in A(W_-)$.

These properties uniquely determine θ_Q and a dimension argument shows that θ_Q is bijective [9]. Now suppose R is a skew symmetric map on W. If $\{e_k\}$ is any complex basis for W and $\{e_k^*\}$ is the corresponding dual basis we define an element of $A^2(W)$ by $\sum_j Re_j \wedge e_j^*$. The reader may check that this does not depend on the choice of the basis $\{e_k\}$ and for brevity we shall often denote the element of $A^2(W)$ by R as well. Let G denote a complex orthogonal and suppose $Q_- G + Q_+$ is invertible on W. Define $R_Q(G) = (G-1)(Q_- G + Q_+)^{-1}$. It is easy to verify that $R_Q(G)^{\tau} = -R_Q(G)$. Our first result in this section is:

THEOREM 2.0. *Suppose* $g \in G(W)$ *and* $Q_- T(g) + Q_+$ *is invertible. Then* $\langle g \rangle_Q \neq 0$ *and*

$$g = \langle g \rangle_Q \theta_Q \exp\{\tfrac{1}{2} R_Q(g)\},$$

where we have written $R_Q(g) \stackrel{\text{def}}{=} R_Q(T(g)) \in \Lambda^2(W)$ *and the exponential is calculated in the Grassmann algebra.*

Proof. The proof can be extracted from Lemma (1.0) and Theorem (1.1) in [15].

Remark. If R is skew symmetric on W and $1 - RQ_-$ is invertible then $G = (1 - RQ_-)^{-1} (1 + RQ_+)$ is complex orthogonal on W and $R = (G - 1)(Q_- G + Q_+)^{-1}$. If $(1 - RQ_-)$ is invertible then it is clear that $G(\lambda) = (1 - \lambda RQ_-)^{-1} (1 + \lambda RQ_+)$ gives a deformation of G to I along any path of values for λ which joins $\lambda = 1$ to $\lambda = 0$ avoiding the finite number of values for λ at which $(1 - \lambda RQ_-)$ is singular. Thus $G = (1 - RQ_-)^{-1} (1 + RQ_+) \in SO_{\mathbb{C}}(W)$. Over the patch in $SO_{\mathbb{C}}(W)$ covered by the R_Q coordinates all the additional information in $\mathrm{Spin}_{\mathbb{C}}(W)$ is carried by $\langle g \rangle_Q$ which we now examine in more detail.

THEOREM 2.1. *Suppose* $g \in G(W)$, *then*

$$\langle g \rangle_Q^2 = nr(g) \det(Q_- T(g) + Q_+).$$

Observe that this theorem implies that $\langle g \rangle_Q \neq 0$ when $Q_- T(g) + Q_+$ is invertible.

We shall not give the most direct derivation of this theorem, which might proceed by a calculation of nr [34I]. Instead we will show that it follows from a more general result that provides an illuminating interpretation of the spinor norm and which will eventually make possible connections with the det* bundle formalism of Segal and Wilson [35]. Let Ω denote, as above, the product of the elements of a basis for W in $\mathscr{C}(W)$ normalized so that $\Omega^2 = 1$. Then

$$x \otimes 1 + \Omega \otimes y \to x \oplus y$$

extends to a Clifford algebra isomorphism from $\mathscr{C}(W) \otimes \mathscr{C}(W)$ onto $\mathscr{C}(W \oplus W)$, where the bilinear form on $W \oplus W$ is $(x_1 \oplus x_2, y_1 \oplus y_2) = (x_1, y_1) + (x_2, y_2)$. We now define a homomorphism λ: $G(W) \to G(W \oplus W)$ by

$$\lambda(g) = \begin{cases} g \otimes g & \text{if } g \text{ is even in } \mathscr{C}(W) \\ i\Omega g \otimes g & \text{if } g \text{ is odd in } \mathscr{C}(W). \end{cases}$$

We leave to the reader the simple check that this is a homomorphism. The factor i in the second part is included to guarantee this. When g is even we have $g\otimes g(x\otimes 1+\Omega\otimes y)(g\otimes g)^{-1} = gxg^{-1}\otimes 1+\Omega\otimes gyg^{-1} = T(g)x\otimes 1+\Omega\otimes T(g)\,y$ (since $\Omega g = g\Omega$). When g is odd we have

$$\begin{aligned}(i\Omega g)&\otimes g(x\otimes 1+\Omega\otimes y)[(i\Omega g)\otimes g]^{-1}\\ &= -gxg^{-1}\otimes 1-\Omega\otimes gyg^{-1} = T(g)x\otimes 1+\Omega\otimes T(g)\,y,\end{aligned}$$

where we used $g\Omega = -\Omega g$. The factor Ω in the second part of the definition of λ was inserted precisely to make this last calculation work out. We have then

$$T(\lambda(g)) = T(g)\oplus T(g).$$

Observe that $Q\oplus Q$ is a skew symmetric involution on $W\oplus W$. Part of our interest in λ is that

$$\langle\lambda(g)\rangle_{Q\oplus Q} = \langle g\rangle_Q^2.$$

To see this note first that the Clifford algebra isomorphism $\mathscr{C}(W\oplus W)\simeq\mathscr{C}(W)\otimes\mathscr{C}(W)$ leads to the following well-known realization of the $Q\oplus Q$ Fock representation on $A(W_+)\otimes A(W_+)$,

$$F_{Q\oplus Q}(x\oplus y) = F_Q(x)\otimes I+F_Q(\Omega)\otimes F_Q(y)$$

(note $F_Q(\Omega) = (-1)^N$ where $N\,|\,A^k(W_+) = k$). It is not hard to check that the $Q\oplus Q$ Fock state is given in this realization by $\langle F_{Q\oplus Q}(g)\,1_Q\otimes 1_Q,\,1_Q^*\otimes 1_Q^*\rangle$. For g even we have $F_{Q\oplus Q}(\lambda(g)) = F_Q(g)\otimes F_Q(g)$ so that $\langle\lambda(g)\rangle_{Q\oplus Q} = \langle F_Q(g)\,1_Q,\,1_Q^*\rangle^2 = \langle g\rangle_Q^2$. For g odd we have $\langle g\rangle_Q = 0$ which implies $\langle\lambda(g)\rangle_{Q\oplus Q} = 0$. Thus $\langle\lambda(g)\rangle_{Q\oplus Q} = \langle g\rangle_Q^2$ is trivially true in this case as well.

The other reason we are interested in λ has to do with its connection with the spinor norm. Observe that $W\oplus W$ has a skew symmetric involution that commutes with orthogonals of the form $G\oplus G$, namely $Q_0 = \left[\begin{smallmatrix}0 & i\\ -i & 0\end{smallmatrix}\right]$. We claim that

$$nr(g) = \langle\lambda(g)\rangle_0,$$

where we have written $\langle\cdot\rangle_0 \overset{\text{def}}{=} \langle\cdot\rangle_{Q_0}$.

To see this easily we first identify $\mathscr{C}(W\oplus W)$ with $\mathscr{C}(W\oplus W^*)$ in such a fashion that Q_0 becomes $\left[\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right]$. Here $W\oplus W^*$ will be used to denote the complex orthogonal space $W\oplus W$ with the symmetric bilinear form $(x_1\oplus x_2,\,y_1\oplus y_2) = (x_1,\,y_2)+(x_2,\,y_1)$. The map by which we do this is the extension of the complex orthogonal $(1/\sqrt{2})\left[\begin{smallmatrix}-i & 1\\ i & 1\end{smallmatrix}\right]$: $W\oplus W\to W\oplus W^*$ to an isomorphism of $\mathscr{C}(W\oplus W)$ with $\mathscr{C}(W\oplus W^*)$. The reader may check that Q_0 is carried into $\left[\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right]$ by this map. Under this isomorphism the element $\lambda(g)$ becomes an element of $G(W\oplus W^*)$ which we continue to denote by $\lambda(g)$ when no confusion seems likely. It is clear that $T(\lambda(g))=T(g)\oplus T(g)\in O_{\mathbb{C}}(W\oplus W^*)$. On $W\oplus W^*$ the complex orthogonals of the form $G\oplus G$ are part of the larger group of all complex orthogonals which commute with $Q_0=\left[\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right]$. It is easily seen that any such orthogonal must have the form $\left[\begin{smallmatrix}X&0\\0&X^{-\tau}\end{smallmatrix}\right]$ where $X\in GL(W)$ (the complex linear invertible maps on W). Let $G(W\oplus W^*)_0$ denote the subgroup of $G(W\oplus W^*)$ whose induced rotations are of the form $\left(\begin{smallmatrix}X&0\\0&X^{-\tau}\end{smallmatrix}\right)$ for $X\in GL(W)$. We will now see that the exact sequence

$$\mathbb{C}^* \longrightarrow G(W\oplus W^*)_0 \xrightarrow{T} GL(W) \longrightarrow 0$$

splits (we write $T(g)=X$ rather than $T(g)=X\oplus X^{-\tau}$). Let $g\in G(W\oplus W^*)_0$ and write $G=T(g)\in GL(W)$. Define $\Gamma(G)=I\oplus G\oplus(G\otimes G)\oplus\cdots\oplus(G\otimes\cdots\otimes G)$ acting on $A(W)$, the Fock space for the Q_0 Fock representation of $\mathscr{C}(W\oplus W^*)$. Then

$$\Gamma(G)\,F_0(x\oplus y)\Gamma(G)^{-1}=F_0(Gx\oplus G^{-\tau}y),$$

where F_0 is the Q_0-Fock representation. Since the Fock representation is irreducible we have

$$F_0(g)=(\text{const.})\Gamma(G).$$

But $\langle\Gamma(G)\,1_{Q_0},1^*_{Q_0}\rangle=1$ so we have

$$F_0(g)=\langle g\rangle_0\Gamma(G).$$

Since $g\to\Gamma(G)$ is clearly a homomorphism it follows that $g\to\langle g\rangle_0$ is a homomorphism. It is now trivial to check that $nr(g)=\langle\lambda(g)\rangle_0$. Both sides are homomorphisms from $G(W)$ into $\mathbb{C}^*$ so it is enough to check this relation for $g=w\in W\subseteq C(W)$. But $\lambda(w)=i\Omega w\oplus w=(i\Omega\otimes w)(w\otimes 1)\sim(0\oplus iw)\cdot(w\oplus 0)$ and

$$\begin{aligned}\langle(0\oplus iw)\cdot(w\oplus 0)\rangle_0&=\tfrac{1}{2}\langle(iw\oplus iw)\cdot(-iw\oplus iw)\rangle_{\left[\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right]}\\&=\tfrac{1}{2}\langle(0\oplus iw)(-iw\oplus 0)\rangle_{\left[\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right]}=\tfrac{1}{2}(w,w)=nr(w).\end{aligned}$$

We also have:

THEOREM 2.2. *Suppose $g\in G(W\oplus W^*)_0$ and Q is a skew symmetric involution on W. Then*

$$\langle g\rangle_{Q\oplus Q}=\langle g\rangle_0\det(Q_-G+Q_+),$$

where $T(g)=G$.

Proof. We use Theorem 2.0 to write

$$g = \langle g \rangle_0 \theta_0 \exp\{\tfrac{1}{2} R_0(g)\},$$

where $\theta_0 = \theta_{Q_0}$ and $R_0(g) = R_{Q_0}(T(g))$. One easily calculates that $R_0(g) = (G - I) \oplus (I - G^{\tau})$ as a map on $W \oplus W^*$. Let $\{e_k\}$ denote a basis for W and $\{e_k^*\}$ the dual basis for W^*. The element of $\Lambda^2(W \oplus W^*)$ which corresponds to $\frac{1}{2}R_0(g)$ is then

$$\sum_k (G - 1)\, e_k \wedge e_k^*.$$

Thus

$$\langle g \rangle_{Q \oplus Q} = \langle g \rangle_0 \left\langle \theta_0 \left\{ \exp \sum_k (G - 1)\, e_k \wedge e_k^* \right\} \right\rangle_{Q \oplus Q}.$$

Let $R = G - 1$, then

$$\theta_0 \left\{ \exp \sum_k R e_k \wedge e_k^* \right\} = 1 + \sum_{l=1}^{N} \sum_k R e_{k_1} \cdots R e_{k_l} e_{k_l}^* \cdots e_{k_1}^*,$$

where the k sum on the right is over all multi-indices k such that $1 \leqslant k_1 < k_2 < \cdots < k_l \leqslant N$ and the terms in the sum are products in $\mathscr{C}(W \oplus W^*)$. Now (see 2.2 in [15])

$$\langle R e_{k_1} \cdots R e_{k_l} \cdots e_{k_l}^* \cdots e_{k_1}^* \rangle_{Q \oplus Q} = Pf \begin{bmatrix} 0 & A \\ -A & 0 \end{bmatrix},$$

where the matrix A is the $l \times l$ matrix given by

$$A = \begin{bmatrix} (Q_- R e_{k_1}, e_{k_l}^*) & \cdots & (Q_- R e_{k_1}, e_{k_1}^*) \\ \vdots & & \vdots \\ (Q_- R e_{k_l}, e_{k_l}^*) & \cdots & (Q_- R e_{k_l}, e_{k_1}^*) \end{bmatrix}.$$

But $Pf \left[\begin{smallmatrix} 0 & A \\ -A & 0 \end{smallmatrix} \right] = (-1)^{l(l-1)/2} \det A$. The sum we wish to calculate is thus

$$1 + \sum_{l=1}^{N} \sum_k \det \begin{bmatrix} (Q_- R e_{k_1}, e_{k_1}^*) & \cdots & (Q_- R e_{k_1}, e_{k_l}^*) \\ \vdots & & \vdots \\ (Q_- R e_{k_l}, e_{k_1}^*) & \cdots & (Q_- R e_{k_l}, e_{k_l}^*) \end{bmatrix},$$

where the sum is over multi-indices $1 \leqslant k_1 < k_2 \cdots < k_l \leqslant N$. This is the Fredholm expansion $\sum_{l=0}^{N} \mathrm{Tr}(\Lambda^l(Q_- R))$ for $\det(1 + Q_- R) = \det(1 + Q_-(G - 1)) = \det(Q_- G + Q_+)$ (see [37]). Q.E.D.

As a corollary we deduce Theorem 2.1:

$$\begin{aligned}\langle g\rangle_Q^2 &= \langle \lambda(g)\rangle_{Q\oplus Q} = \langle \lambda(g)\rangle_0 \det(Q_- G + Q_+)\\ &= nr(g)\det(Q_- G + Q_+).\end{aligned}$$

We come now to a result of S.M.J. [8] which will be one of the two principal tools we use to construct infinite dimensional spin groups. Let Q and Q' denote two skew symmetric involutions on W.

THEOREM 2.3. *Suppose $g \in G(W)$ and $\langle g\rangle_Q \neq 0$. Then*

$$\langle g\rangle_{Q'}/\langle g\rangle_Q = \mathrm{Pf}\begin{bmatrix} R_Q(g)/2 & I \\ -I & Q-Q' \end{bmatrix}\Big/\mathrm{Pf}\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}, \tag{2.1}$$

where $R_Q(g) = (T(g)-I)(Q_- T(g) + Q_+)^{-1}$, the matrices I are identity matrices on W, and Pf *is the Pfaffian.*

Remark. Pfaffians are, of course, only defined for skew symmetric matrices. The matrices for $R_Q(g)$, Q, and Q' will all be skew symmetric relative to any *self-dual* basis for W. This gives the sense in which formula (2.1) should be understood.

Proof. Observe that as a consequence of Theorem 2.1

$$\frac{\langle g\rangle_{Q'}^2}{\langle g\rangle_Q^2} = \frac{\det(Q'_- T(g) + Q'_+)}{\det(Q_- T(g) + Q_+)} = \det(1 + (Q-Q')\, R_Q(g)/2).$$

The last equality is presented with more details in Lemma 2.2 of [15]. Suppose A and B are $n \times n$ complex skew symmetric matrices. Then

$$\left[\mathrm{Pf}\begin{pmatrix} A & I \\ -I & B \end{pmatrix}\right]^2 = \det\begin{bmatrix} A & I \\ -I & B \end{bmatrix} = \det(I + BA).$$

Specializing this last result to $A = R_Q(g)/2$ and $B = Q - Q'$ and observing that $\mathrm{Pf}[{}^{\ 0}_{-I}\ {}^{I}_{0}] = \pm 1$ we find

$$\langle g\rangle_{Q'}/\langle g\rangle_Q = \pm\left(\mathrm{Pf}\begin{bmatrix} R_Q(g)/2 & I \\ -I & (Q-Q') \end{bmatrix}\Big/\mathrm{Pf}\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}\right).$$

There is a possible sign ambiguity in this result which we resolve in the following manner. Recall from Theorem 2.0 that $g = \langle g\rangle_Q \theta_Q \exp[R_Q(g)/2]$ and define $g(\lambda) = \langle g\rangle_Q \theta_Q \exp[\lambda R_Q(g)/2]$. Then $g(\lambda)$ is a polynomial function of λ with values in $G(W)$ except at the finite number of values for λ at which $1 - \lambda R_Q(g)\, Q_-$ is singular. The functions

$$\lambda \to \frac{\langle g(\lambda)\rangle_{Q'}}{\langle g(\lambda)\rangle_Q} = \langle \theta_Q \exp[\lambda R_Q(g)/2]\rangle_{Q'}$$

and

$$\lambda \to \mathrm{Pf}\begin{bmatrix} \lambda R_Q(g)/2 & I \\ -I & Q-Q' \end{bmatrix} \Big/ \mathrm{Pf}\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

are thus two polynomials in λ which differ at most by a sign. However, both functions are 1 at $\lambda = 0$ and it follows that they agree for all λ, in particular for $\lambda = 1$. Q.E.D.

A second result we require is:

THEOREM 2.4. *Suppose* $g_j \in G(W)$ $(j=1,2)$, $\langle g_j \rangle_Q \neq 0$ $(j=1,2)$, *and* $T(g_j) = \left[\begin{smallmatrix} A_j & B_j \\ C_j & D_j \end{smallmatrix}\right]$ *relative to the splitting of* W *determined by* Q. *Then*

$$\langle g_2 g_1 \rangle_Q = \langle g_2 \rangle_Q \langle g_1 \rangle_Q \left(\mathrm{Pf}\begin{bmatrix} B_1 D_1^{-1} & I \\ -I & D_2^{-1} C_2 \end{bmatrix} \Big/ \mathrm{Pf}\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \right). \tag{2.2}$$

Before we prove this we make some remarks about how (2.2) is to be understood. As noted earlier the condition $\langle g_j \rangle_Q \neq 0$ implies $Q_- T(g_j) + Q_+$ is invertible and hence that D_j is invertible. The condition $G_1^\tau G_1 = I$ implies $D_1^\tau B_1 + B_1^\tau D_1 = 0$ or $(B_1 D_1^{-1})^\tau = -B_1 D_1^{-1}$. Thus $B_1 D_1^{-1}$ is skew-symmetric from W_- to W_+. By composing $B_1 D_1^{-1}$ with Q_- we may regard $B_1 D_1^{-1}$ as a map on all of W. It remains skew-symmetric on all of W and this is the sense in which it is to be understood in (2.2). In a similar fashion $G_2 G_2^\tau = I$ implies $D_2^{-1} C_2$ is skew-symmetric as a map from W_+ to W_-. As above one should think of $D_2^{-1} C_2 Q_+$ in (2.2) where for brevity $D_2^{-1} C_2$ appears. The map

$$\begin{bmatrix} B_1 D_1^{-1} Q_- & I \\ -I & D_2^{-1} C_2 Q_+ \end{bmatrix}$$

on $W \oplus W$ has a skew symmetric matrix relative to any basis for $W \oplus W$ induced by a self-dual basis on W. This is the matrix which one takes the Pfaffian of in (2.2).

Proof. As in the preceding theorem we begin with a result for $\langle g_2 g_1 \rangle_Q^2$. Theorem 2.1 implies $\langle g_2 g_1 \rangle_Q^2 = nr(g_2 g_1) \det D_Q(g_2 g_1)$ where $D_Q(g_2 g_1) = D_Q(T(g_2 g_1))$. Multiplying the matrices for $T(g_2)$ and $T(g_1)$ one finds $D_Q(g_2 g_1) = C_2 B_1 + D_2 D_1$. Thus

$$\begin{aligned} \langle g_2 g_1 \rangle^2 &= nr(g_2 g_1) \det(C_1 B_1 + D_2 D_1) \\ &= nr(g_2) \det(D_2)\, nr(g_1) \det(D_1) \det(I + D_2^{-1} C_2 B_1 D_1^{-1}) \\ &= \langle g_2 \rangle_Q^2 \langle g_1 \rangle_Q^2 \left(\mathrm{Pf}\begin{bmatrix} B_1 D_1^{-1} & I \\ -I & D_2^{-1} C_2 \end{bmatrix} \Big/ \mathrm{Pf}\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \right)^2, \end{aligned}$$

where we used Theorem 2.1 and $\det(I+BA)=(\operatorname{Pf}[\begin{smallmatrix} A & I \\ -I & B\end{smallmatrix}])^2$ again. Taking square roots we have

$$\langle g_2 g_1\rangle_Q = \pm\langle g_2\rangle_Q\langle g_1\rangle_Q\left(\operatorname{Pf}\begin{bmatrix} B_1D_1^{-1} & I \\ -I & D_2^{-1}C_2\end{bmatrix}\Big/\operatorname{Pf}\begin{bmatrix} 0 & I \\ -I & 0\end{bmatrix}\right).$$

To resolve the sign ambiguity we introduce the deformation $g_1(\lambda)=\langle g_1\rangle_Q\theta_Q\exp[\lambda R_Q(g)/2]$ which takes values in $G(W)$ except at those values of λ for which $(1-\lambda R_Q(g)\,Q_-)$ is singular. Since $B(g_1(\lambda))\,D(g_1(\lambda))^{-1}=Q_-\,R_Q(g_1(\lambda))\,Q_+=\lambda Q_-\,R_Q(f_1)\,Q_+=\lambda B_1D_1^{-1}$ we have

$$\langle g_2 g_1(\lambda)\rangle = \pm\langle g_2\rangle_Q\langle g_1\rangle_Q\left(\operatorname{Pf}\begin{bmatrix} \lambda B_1D_1^{-1} & I \\ -I & C_2D_2^{-1}\end{bmatrix}\Big/\operatorname{Pf}\begin{bmatrix} 0 & I \\ -I & 0\end{bmatrix}\right) \tag{2.3}$$

except possibly at the finite number of values λ where $\det(1-\lambda R_Q(g_1)\,Q_-)=0$. However, both sides of (2.3) are polynomials in λ; the choice of $\pm$ sign in (2.3) is thus fixed by its value at $\lambda=0$ where it is $+$. This proves (2.2). Q.E.D.

Before we turn to infinite dimensional considerations we offer the following translation of the Pfaffians which appear in Theorems 2.3 and 2.4.

LEMMA 2.5. *Suppose A and B are $n\times n$ complex skew symmetric matrices. Let $\gamma\colon [0,1]\to\mathbb{C}$ be a smooth simple curve with $\gamma(0)=0$, $\gamma(1)=1$ and such that γ does not pass through any of the points λ where $(1+\lambda BA)$ is singular. Then*

$$\operatorname{Pf}\begin{bmatrix} A & I \\ -I & B\end{bmatrix}\Big/\operatorname{Pf}\begin{bmatrix} 0 & I \\ -I & 0\end{bmatrix}=\exp\frac{1}{2}\int_\gamma \operatorname{Tr}((1+\lambda BA)^{-1}\,BA)\,d\lambda.$$

Proof. The path γ is contained in a simply connected domain D which does not contain any λ for which $\det(1+\lambda BA)=0$. Thus there exists a single valued analytic branch of $\log\det(1+\lambda BA)$ for $\lambda\in D$ which we normalize to 0 at $\lambda=0$. Suppose $\lambda\in D$ and γ_λ is a smooth curve which joins 0 to λ in D. Then $\exp\frac{1}{2}\int_{\gamma_\lambda}(d/d\lambda)\log\det(1+\lambda BA)\,d\lambda$ gives an analytic square root for $\det(1+\lambda BA)$ which agrees with $\operatorname{Pf}[\begin{smallmatrix} A & I \\ -I & B\end{smallmatrix}]/\operatorname{Pf}[\begin{smallmatrix} 0 & I \\ -I & 0\end{smallmatrix}]$ at $\lambda=0$. The two analytic square roots therefore agree in all of D and hence at $\lambda=1$. At points where $(1+\lambda BA)$ is invertible

$$\frac{d}{d\lambda}\log\det(1+\lambda BA)=\operatorname{Tr}((1+\lambda BA)^{-1}\,BA)$$

and this finishes the proof of this lemma. Q.E.D.

We conclude this section with some remarks concerning the transition to infinite dimensional W. Our construction of an infinite dimensional spin group will depend on a Hilbert space structure for W and the splitting induced on W by a skew involution Q. It will facilitate our work considerably to ensure that these two ingredients fit together well. By this we mean that the splitting $W_+ + W_-$ induced by Q is an orthogonal direct sum with respect to the Hilbert space structure. This can always be arranged in the finite dimensional cases as we now demonstrate. Let W be a complex vector space with a non-degenerate symmetric bilinear form $(\cdot,\cdot)$. Let $W = W_+ + W_-$ be the isotropic splitting of W induced by the skew symmetric involution Q. Let $\{e_1, ..., e_n\}$ be a basis for W_+ and let $\{e_1^*, ..., e_n^*\}$ be the corresponding dual basis for W_-. Define a conjugate linear map from W onto W by

$$P\left(\sum_i (a_i e_i + b_i e_i^*)\right) = \sum_i (\bar{a}_i e_i^* + \bar{b}_i e_i).$$

It is clear that the Hermitian inner product defined on W by $\langle x, y\rangle = (x, Py)$ is positive definite. Furthermore since Q obviously anticommutes with P we have

$$\langle Qx, y\rangle = (Qx, Py) = -(x, QPy) = (x, PQy) = \langle x, Qy\rangle$$

so that Q is symmetric. Our starting point in the infinite dimensional case will be to build in this relationship. W will be a complex Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and a bilinear form $(\cdot,\cdot)$ obtained from $\langle\cdot,\cdot\rangle$ by a conjugation P so that $(x,y) = \langle x, Py\rangle$. The skew involution Q which determines the splitting will be assumed Hermitian symmetric with $QP + PQ = 0$.

Before we proceed to the infinite dimensional case we rework the Fock space construction to incorporate this additional structure. The space W_+ is now a finite dimensional Hilbert space. Let $\{e_k\}$ be an orthonormal basis for W_+. We endow $A(W_+)$ with a finite dimensional Hilbert space structure by fixing the inner product on $A^l(W_+)$ so that $e_{i_1} \wedge \cdots \wedge e_{i_l}$ $(i_1 < i_2 \cdots < i_l)$ is an orthonormal basis. We leave it to the reader to check that this Hilbert space structure does not depend on the original choice of an orthonormal basis $\{e_k\}$ for W_+. For $x \in W_+$ define

$$a^*(x) = C(x)$$

and let $a(x) = (a^*(x))^*$ where the adjoint $*$ is taken with respect to the Hermitian inner product on $A(W_+)$. It is not hard to check that $a(x) = C^\tau(Px)$. Thus $F_Q(x \oplus y) = C(x) + C^\tau(y) = a^*(x) + a(\bar{y})$ for $x \in W_+$, $y \in W_-$ and $\bar{y} = Py$. The conjugation P on W induces a conjugation on

$\mathscr{C}(W)$ in an obvious fashion. We write $Pg = \bar{g}$ for $g \in \mathscr{C}(W)$. Combining this conjugation with the transpose, one has the $*$ involution $g \to \bar{g}^{\tau} \stackrel{\text{def}}{=} g^*$. There are two properties of this involution which will be important for us. If $g \in \mathrm{Spin}_{\mathbb{C}}(W)$ then so is g^*. To see this observe first that if $G \in SO_{\mathbb{C}}(W)$ then $G^* = PG^{\tau}P$ is in $SO_{\mathbb{C}}(W)$ since $(PG^{\tau}Px, PG^{\tau}Py) = \overline{(G^{\tau}Px, G^{\tau}Py)} = \overline{(Px, Py)} = (x, y)$. Now take conjugates then transposes of both sides of $gxg^{-1} = T(g)x$ to get $g^{*-1}\bar{x}g^* = \overline{T(g)}\,\bar{x}$ (where $\overline{T(g)} = PT(g)P$). From this it follows that $g^*xg^{*-1} = \overline{T(g)}^{-1}\,x = \overline{T(g)}^{\tau}\,x = T(g)^*\,x$. Thus $g^* \in \mathrm{Spin}_{\mathbb{C}}(W)$ and $T(g^*) = T(g)^*$. The second property that will be useful is that $\mathscr{C}(W) \ni g \to F_Q(g)$ is a $*$ algebra homomorphism. That is $F_Q(g^*) = F_Q(g)^*$ where the adjoint on the right is relative to the Hermitian inner product on $A(W_+)$. It is clearly enough to check this for $g = w \in W$ where we have

$$\begin{aligned} F_Q(w^*) &= F_Q(\bar{w}) = F_Q(\bar{w}_- + \bar{w}_+) = a^*(\bar{w}_-) + a(w_+) \\ &= (a^*(w_+) + a(\bar{w}_-))^* = F_Q(w)^*. \end{aligned}$$

When W is infinite dimensional, W_+ is an infinite dimensional Hilbert space with inner product $\langle \cdot,\cdot \rangle$. It is customary to define annihilation and creation operators $a(x)$ and $a^*(x)$ on $A(W_+)$ so that

$$\begin{aligned} a^*(x)\,a(y) + a(y)\,a^*(x) &= \langle x, y \rangle I \\ a(x)a(y) + a(y)a(x) &= 0 \end{aligned}$$

for $x, y \in W_+$ and $a(x)1 = 0$ for $x \in W_+$. The Q-Fock representation of the Clifford relations is then given by

$$F_Q(w) = a^*(w_+) + a(\bar{w}_-).$$

For details of this construction we refer the reader to [29].

3. The Infinite Spin Group $\mathrm{Spin}_Q(W)$

In this section W is infinite dimensional, $\mathscr{C}_0(W)$ is the algebraic Clifford algebra consisting of finite sums of finite products from W. The Clifford group $G_0(W)$ is the collection of $g \in \mathscr{C}_0(W)$ such that $gxg^{-1} = Gx$ for $x \in W$ and G a complex orthogonal on W. As before elements in $G_0(W)$ will be either even or odd in $\mathscr{C}_0(W)$ and we define

$$gxg^{-1} = \begin{cases} T(g)x, & g \text{ even in } \mathscr{C}_0(W) \\ -T(g)x, & g \text{ odd in } \mathscr{C}_0(W). \end{cases}$$

This gives rise to an exact sequence,

$$\mathbb{C}^* \longrightarrow G_0(W) \xrightarrow{T} O_0(W) \longrightarrow 0,$$

where $O_0(W) = \{G \mid G^\tau = G^{-1}$ and $G - I$ is finite rank$\}$.

Our first order of business in this section is to translate Theorems 2.5 and 2.6 into the infinite dimensional setting. In this section P will denote a fixed conjugation on W which gives rise to the distinguished bilinear form on W and Q will denote a self-adjoint involution on W which anti-commutes with P. We begin with a description of the family of functionals for which we require a generalization of Theorem 2.6. If Q' is an involution on W such that $(Q')^\tau = -Q'$ and $Q' - Q$ is finite rank then we will say that Q' is a *Q-finite involution.* Such operators will play an important auxiliary role in what follows. We wish to define the Q'-functional on $\mathscr{C}_0(W)$. If $g \in \mathscr{C}_0(W)$ then the following lemma (3.0) gives us a finite dimensional subspace W_f invariant under $P, Q,$ and Q' such that $g \in \mathscr{C}(W_f)$. We may then define $\langle g \rangle_{Q'}$ by simply regarding g as an element of $\mathscr{C}(W_f)$.

LEMMA 3.0. *Let Q be a self-adjoint involution which anti-commutes with the conjugation P. Let Q_k $(k = 1, 2)$ be a Q-finite involution on W. If W_0 is a finite dimensinal subspace of W, then there exists a finite dimensional subspace $W_f \supseteq W_0$ with W_f invariant under $P, Q,$ and Q_k $(k = 1, 2)$.*

Proof. Since P and Q anti-commute and are involutions it is clear that $W_0 + QW_0 + PW_0$ is a finite dimensional subspace containing W_0 and invariant under P and Q. We may therefore assume that W_0 is invariant under P and Q. With this understood define

$$\begin{aligned} W_f = W_0 &+ Q_1 W_0 + Q_2 W_0 + (Q - Q_1) W + (Q - Q_2) W \\ &+ (Q_1 - Q_2) W + P(Q - Q_1) W + P(Q - Q_2) W + P(Q_1 - Q_2) W. \end{aligned}$$

Since $Q - Q_k$ is finite rank it is clear that W_f is finite dimensional. A routine calculation shows that W_f is invariant under $P, Q,$ and Q_k $(k = 1, 2)$. Q.E.D.

To see that the definition of $\langle g \rangle_{Q'}$ does not depend on the particular choice of W_f we use:

LEMMA 3.1. *Let W_f and W'_f be finite dimensional P-invariant subspaces of W. Then*

$$\mathscr{C}(W_f) \cap \mathscr{C}(W'_f) = \mathscr{C}(W_f \cap W'_f).$$

Proof. Let $V = W_f + W'_f$. Since V is P-invariant the bilinear form on V is non-degenerate. Let $v_1, v_2, ..., v_n$ be any basis for V. Then the products $v_{i_1} \cdots v_{i_k}$ with $1 \leqslant i_1 < i_2 \cdots < i_k \leqslant n$ are a basis for $\mathscr{C}(V)$. It is clear that they span $\mathscr{C}(V)$, and since $\mathscr{C}(V)$ has dimension 2^n they must be linearly independent. Let $v_1, ..., v_k$ be a basis for $W_f \cap W'_f$, let $v_{k+1}, ..., v_l$ be a basis for a complement of $W_f \cap W'_f$ in W_f, and let $v_{l+1}, ..., v_n$ be a basis for a complement of $W_f \cap W'_f$ in W'_f. Then $\{v_1, ..., v_n\}$ is a basis for V and by uniqueness for expansion in a basis any element in $\mathscr{C}(W_f) \cap \mathscr{C}(W'_f) \subseteq \mathscr{C}(V)$ must be a sum $\sum_{1 \leq i_1 < \cdots < i_j \leq k} a_{i_1 \cdots i_j} v_{i_1} \cdots v_{i_j} \in \mathscr{C}(W_f \cap W'_f)$. Q.E.D.

Thus if $g \in \mathscr{C}(W_f) \cap \mathscr{C}(W'_f)$ we expand g in terms of basis for $W_f \cap W'_f$ and we conclude that the Q' functional of g does not depend on whether we regard g as an element of $\mathscr{C}(W_f)$ or $\mathscr{C}(W'_f)$. We now translate Theorem 2.6.

THEOREM 3.2. *Suppose* $g \in G_0(W)$. *Let* Q_k $(k=1,2)$ *be* Q*-finite involutions on* W. *Suppose* $\langle g \rangle_{Q_k} \neq 0$ $(k=1,2)$ *and let* $R_1(g) \overset{\text{def}}{=} (T(g)-I)(Q_1^- T(g) + Q_1^+)^{-1}$. *Let* γ *denote a smooth simple path in* $\mathbb{C}$ *joining* 0 *to* 1 *which does not pass through any values* $\lambda \in \mathbb{C}$ *for which* $(1+\lambda(Q_1 - Q_2) R_1(g)/2)$ *is singular. Then*

$$\frac{\langle g \rangle_{Q_2}}{\langle g \rangle_{Q_1}} = \exp\left\{\frac{1}{2}\int_\gamma \operatorname{Tr}((1+\lambda A)^{-1} A)\, d\lambda\right\}, \tag{3.1}$$

where $A = (Q_1 - Q_2) R_1(g)/2$.

Proof. We may apply Theorem 2.6 and Lemma 2.8 directly once we know that $g \in \mathscr{C}(W_f)$ where W_f is a finite dimensional subspace invariant under P, Q, and Q_k $(k=1,2)$ (invariance under P ensures the non-degeneracy of the inner product, invariance under Q and P ensures that W_f is even dimensional). The existence of such a W_f is guaranteed by Lemma 2.1. In the formula which results from the application of Theorem 2.6 and Lemma 2.8 the operator $A = (Q_1 - Q_2) R_1(g)/2$ is to be regarded as a map on W_f. If W_f does not contain the range of $R_1(g)$ then we may remedy this by taking the sum of W_f with the (finite dimensional) range of $R_1(g)$ and close up the resulting subspace under P, Q, and Q_k $(k=1,2)$ using Lemma 2.1. We therefore assume that the range of $R_1(g)$ is contained in W_f. Let $W_f^\perp$ denote the Hilbert space orthogonal complement to W_f. Because W_f is P invariant it follows that W_f is also orthogonal to W_f relative to the bilinear pairing $\langle \cdot, P \cdot \rangle$. Since $R_1(g)^\tau = -R_1(g)$ it follows that $R_1(g)|_{W_f^\perp} = 0$. From this it is clear that the "A" in formula (3.1) may be regarded as a map on all of W without changing the right hand side. Q.E.D.

Next we extend Theorem 2.7. First we introduce the groups $SO_0(W) =$ the subgroup of $O_0(W)$ with determinant 1 and

$$\operatorname{Spin}_0(W) = \{g \in G_0(W) \mid T(g) \in SO_0(W) \text{ and } nr(g) = 1\}.$$

THEOREM 3.3. *Suppose* $g_j \in \mathrm{Spin}_0(W)$ $(j=1,2)$, $\langle g_j \rangle_Q \neq 0$ $(j=1,2)$ *and let* $T(g_j) = \left[\begin{smallmatrix} A_j & B_j \\ C_j & D_j \end{smallmatrix}\right]$ *relative to the splitting of* W *determined by* Q. *Then*

$$\langle g_2 g_1 \rangle_Q = \langle g_2 \rangle_Q \langle g_1 \rangle_Q \exp \tfrac{1}{2} \int_\gamma \mathrm{Tr}((1+\lambda B)^{-1} BA)\, d\lambda, \tag{3.2}$$

where $A = B_1 D_1^{-1}$, $B = D_2^{-1} C_2$, *and* γ *is a smooth simple curve which joins* 0 *to* 1 *in* $\mathbb{C}$ *and avoids the values of* λ *for which* $(1+\lambda BA)$ *fails to be invertible.*

Proof. Let W_f be the sum of the ranges of $R_Q(g_j)$ $(j=1,2)$ closed up under the action of P and Q as in Lemma (2.1). Then Theorem 2.3 applies since $\langle g_j \rangle_Q \neq 0$ for $j=1,2$ and it follows that $g_j \in \mathscr{C}(W_f) \subseteq \mathscr{C}_0(W)$. Theorem 2.6 and Lemma 2.8 apply and give the conclusion of the theorem except that $B_1 D_1^{-1}$ and $D_2^{-1} C_2$ are to be regarded as maps defined on W_f (after composition with Q_- and Q_+, respectively). Observe however that since

$$R(g_j) = \begin{pmatrix} D_j^{-\tau} - 1 & B_j D_j^{-1} \\ D_j^{-1} C_j & 1 - D_j^{-1} \end{pmatrix}$$

and W_f is invariant under $Q_\pm$ and $R(g_j)$ $(j=1,2)$ it follows that W_f contains the range of $B_j D_j^{-1}$ on W_- and the range of $D_j^{-1} C_j$ on W_+. Since both $B_j D_j^{-1}$ and $D_j^{-1} C_j$ are skew symmetric with respect to the bilinear pairing it follows that they vanish identically on $Q_- W_f^\perp$ and $Q_+ W_f^\perp$, respectively, where $W_f^\perp$ denotes the Hilbert space orthogonal complement of W_f as before. It follows that BA in (3.2) may be regarded as a map on all of W_- without changing the right hand side of (3.2). Q.E.D.

Let $O_Q(W)$ denote the group of complex orthogonals G on W with matrices $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ relative to the splitting of W obtained from Q such that A and D are trace class perturbations of the identity on W_+ and W_-, respectively, and B and C are Schmidt class operators. Let $SO_Q(W)$ denote the connected component of the identity in $O_Q(W)$ (i.e., those elements $G \in O_Q(W)$ with $\ker(D(G))$ even dimensional). The topology on $SO_Q(W)$ is given by trace norm convergence on the diagonal and Schmidt norm convergence on the off diagonal.

Let $g, h \in \mathrm{Spin}_0(W)$ and define a metric on $\mathrm{Spin}_0(W)$ by $d_Q(g,h) = \|T(g) - T(h)\|_{1,2} + \|F_Q(g)\, 1_Q - F_Q(h)\, 1_Q\|$ where for $X = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ on $W_+ \oplus W_-$ we define $\|X\|_{1,2} = \|\left[\begin{smallmatrix} A & 0 \\ 0 & D \end{smallmatrix}\right]\|_1 + \|\left[\begin{smallmatrix} 0 & B \\ C & 0 \end{smallmatrix}\right]\|_2$ with $\|\cdot\|_1 =$ trace norm and $\|\cdot\|_2 =$ Schmidt norm. Much of the rest of this section will be devoted to a proof of the following result:

THEOREM 3.4. *The closure of* $\mathrm{Spin}_0(W)$ *in the* d_Q *metric is a continuous group* $\mathrm{Spin}_Q(W)$. *The homomorphism* $T\colon \mathrm{Spin}_0(W) \to SO_0(W)$ *extends to* *a homomorphism* $T\colon \mathrm{Spin}_Q(W) \to SO_Q(W)$ *for which one has the exact sequence*

$$\mathbb{Z}/2\mathbb{Z} \longrightarrow \mathrm{Spin}_Q(W) \xrightarrow{T} SO_Q(W) \longrightarrow 0.$$

For the proofs it is inconvenient to use the metric d_Q directly. It will be simpler to translate d_Q convergence into convergence for the induced rotations plus control over the "sheet" on which the elements in $\mathrm{Spin}_0(W)$ lie over $SO_0(W)$. The Q-finite involutions are useful in this regard as the following proposition indicates.

PROPOSITION 3.5. *A sequence* $g_n \in \mathrm{Spin}_0(W)$ *is a* d_Q *Cauchy sequence if and only if* $T(g_n)$ *converges in* $SO_Q(W)$ *and there exists a* Q*-finite involution* Q' *such that* $\langle g_n \rangle_{Q'}$ *converges to a non-zero value as* $n \to \infty$.

Proof. Suppose first that $G_n \stackrel{\text{def}}{=} T(g_n)$ converges in $SO_Q(W)$ and that there exists a Q-finite involution Q' such that $\langle g_n \rangle_{Q'}$ converges to a non-zero value. We want to prove that $F_Q(g_n)\, 1_Q$ converges in $A(W_+)$. We consider two cases, the case where $D_Q(G)$ is invertible and the case where $D_Q(G)$ is singular. To begin suppose $D_Q(G)$ is invertible. Because $D_Q(G_n)$ converges to $D_Q(G)$ in trace norm it follows that $D_Q(G_n)$ will be invertible for all sufficiently large n. Thus Theorem 3.2 applies and we have

$$\langle g_n \rangle_Q = \langle g_n \rangle_{Q'} \exp\left\{ -\frac{1}{2} \int_\gamma \mathrm{Tr}((1+\lambda A_n)^{-1} A_n)\, d\lambda \right\}, \tag{3.3}$$

where $A_n = (Q - Q')\, R_Q(G_n)/2$ and γ is a simple closed path in $\mathbb{C}$ joining 0 to 1 and avoiding those values of λ for which $(1 + \lambda A_n)$ is singular. Observe that A_n converges in trace norm to $A = (Q - Q')\, R_Q(G)/2$ since $Q - Q'$ is finite rank. For any compact operator A the operator $(1 + \lambda A)$ is singular for only finitely many values of λ in any fixed compact region in $\mathbb{C}$. It is thus possible in our case to choose a smooth simple path γ joining 0 to 1 which avoids the singular values for $(1 + \lambda A)$. The path γ is separated from the value of λ for which $(1 + \lambda A)$ is singular by a finite distance and since A_n converges to A in uniform norm it follows that $(1 + \lambda A_n)^{-1}$ converges to $(1 + \lambda A)^{-1}$ in the uniform norm uniformly for λ in the image of the path γ. The product of a sequence of operators which converges in uniform norm with a sequence that converges in trace norm also converges in trace norm. Thus by dominated convergence $\int_\gamma \mathrm{Tr}((1+\lambda A_n)^{-1} A_n)\, d\lambda$ converges to $\int_\gamma \mathrm{Tr}((1+\lambda A)^{-1} A)\, d\lambda$. Since we are assuming $\langle g_n \rangle_{Q'}$ converges it follows from (3.3) that $\langle g_n \rangle_Q$ also converges. Consulting Theorem 1.1 we see that $\langle g_n \rangle_Q^2 = \det(D_Q(G_n))$. Since $D_Q(G_n)$ converges in trace norm to $D_Q(G)$ and the determinant is continuous in the trace norm it follows that

$\lim_n \langle g_n \rangle_Q^2 = \det D_Q(G) \neq 0$ (since we are supposing $D_Q(G)$ is invertible). It follows from this that $\lim_n \langle g_n \rangle_Q \neq 0$.

Now we can calculate

$$\begin{aligned}\|F_Q(g_n)\,1_Q - F_Q(g_m)\,1_Q\|^2 \\ = \langle g_n^* g_n \rangle_Q + \langle g_m^* g_m \rangle_Q - \langle g_n^* g_m \rangle_Q - \langle g_m^* g_n \rangle_Q. \qquad (3.4)\end{aligned}$$

We will show that $F_Q(g_n)\,1_Q$ is a Cauchy sequence by proving that the Q functionals on the right hand side all approach the same limit as m and n get large. Let $\alpha = m$ or n and $\beta = m$ or n; then since $\langle g_\beta^* \rangle = \overline{\langle g_\beta \rangle_Q} \neq 0$ for β sufficiently large, it follows that Theorem 3.3 applies and we have

$$\langle g_\beta^* g_\alpha \rangle = \langle g_\beta^* \rangle_Q \langle g_\alpha \rangle_Q \exp\left\{\frac{1}{2}\int_\gamma \mathrm{Tr}[(1+\lambda Z_\beta^* Z_\alpha)^{-1} Z_\beta^* Z_\alpha]\,d\lambda\right\},$$

where $Z_\alpha = B(g_\alpha)\,D(g_\alpha)^{-1}$ and $Z_\beta^* = D(g_\beta)^{*-1}\,B(g_\beta)^* = D(g_\beta^*)^{-1}\,C(g_\beta^*)$. The path γ is a smooth simple curve joining 0 to 1 in $\mathbb{C}$ which avoids those values of λ for which $(1+\lambda Z_\beta^* Z_\alpha)$ is singular. Let $Z = B(G)\,D(G)^{-1}$. Then since Z_α is close to Z in Schmidt norm for large α it follows that $Z_\beta^* Z_\alpha$ can be made arbitrarily close to Z^*Z in trace norm by choosing α and β sufficiently large. If we choose a path γ which joins 0 to 1 and avoids those λ for which $1+\lambda Z^*Z$ is singular then as above the curve γ will not pass through the inverse spectral values of $-Z_\beta^* Z_\alpha$ for all α, β sufficiently large. Let $a = \lim_n \langle g_n \rangle_Q$. Then for large α, β the value of $\langle g_\beta^* g_\alpha \rangle_Q$ is arbitrarily close to

$$\bar{a}a \exp\left\{\frac{1}{2}\int_\gamma \mathrm{Tr}((1+\lambda Z^*Z)^{-1} Z^*Z)\,d\lambda\right\},$$

where $Z = B(G)D(G)^{-1}$ and the curve γ is a smooth simple curve joining 0 to 1 in $\mathbb{C}$ without passing through the singular points of $(1+\lambda Z^*Z)$. Thus the terms on the right hand side of (3.4) cancel for large m, n and we have shown that $F_Q(g_n)\,1_Q$ converges.

Next we consider the case where $D_Q(G)$ is not invertible. By Lemma 1.2 there is an $H = [\begin{smallmatrix}1 & 0\\ b & 1\end{smallmatrix}]$ in $SO_0(W)$ such that $D_Q(HGH^{-1})$ is invertible. There is an $h \in G_0(W)$ such that $T(h) = H$. In fact $R_Q(H) = (\begin{smallmatrix}0 & 0\\ b & 0\end{smallmatrix})$ is finite rank and $F_Q(\theta_Q \exp \frac{1}{2}R_Q(H))$ implements H in the Q-Fock representation. Choose a constant c so that $nr(c\theta_Q \exp\frac{1}{2}R_Q(H)) = 1$ and let $h = c\theta_Q \exp\frac{1}{2}R_Q(H) \in \mathrm{Spin}_0(W)$. Consider the sequence $g_n' = hg_nh^{-1}$. We will show that $\langle g_n' \rangle_Q$ converges to a non-zero value. Since it is already clear that $T(g_n') = HT(g_n)H^{-1}$ converges in $SO_Q(W)$ we may infer from the earlier developments in this proof that g_n' is a d_Q Cauchy sequence. But $F_Q(g_n')\,1_Q = F_Q(h)\,F_Q(g_n)\,F_Q(h^{-1})\,1_Q = c^{-1}F_Q(h)\,F_Q(g_n)\,1_Q$ since $F_Q(h)\,1_Q = c\,1_Q$ and because

$F_Q(h)$ is invertible it follows that $F_Q(g_n)\,1_Q$ converges. Thus we will finish this part of the proof if we can show that $\langle g'_n\rangle_Q$ converges (it will automatically converge to a non-zero value if it converges at all since $\lim_n\langle g'_n\rangle_Q^2 = \lim_n \det D_Q(g'_n) = \det D_Q(HGH^{-1}) \neq 0$). Consider the functional $\mathscr{C}_0(W)\ni X\to\langle F_Q(h)\,F_Q(X)\,F_Q(h^{-1})\,1_Q, 1_Q\rangle$. It is clear that this functional is the composition of the Q-Fock state on $\mathscr{C}_0(W)$ with the automorphism of $\mathscr{C}_0(W)$ induced by $T(h)$. The two point function is $\langle F_Q(h)\, F_Q(w_1 w_2)\, F_Q(h^{-1})\, 1_Q,\, 1_Q\rangle = \langle T(h)\,w_1\cdot T(h)\,w_2\rangle_Q = \langle Q_-T(h)\,w_1, \overline{T(h)\,w_2}\rangle = \langle T(h)^{-1}\,Q_-\,T(h)\,w_1, \bar{w}_2\rangle = \langle Q'_-\,w_1, \bar{w}_2\rangle$. Where $Q'=\begin{pmatrix}1 & 0\\ -b & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\ b & 1\end{pmatrix}=\begin{pmatrix}1 & 0\\ -2b & -1\end{pmatrix}$. This is a Q-finite involution and we have

$$\langle g'_n\rangle_Q=\langle g_n\rangle_{Q'}.$$

But $\langle g_n\rangle_{Q_1}$ is known to converge to a non-zero value for *some* Q-finite involution Q_1. If we combine this with Theorem 3.2 as was done above we see that $\langle g_n\rangle_{Q'}=\langle g'_n\rangle_Q$ converges. This finishes the proof in one direction.

Now suppose g_n is a d_Q Cauchy sequence. Let $G=\lim_n T(g_n)$ and choose $H=\begin{pmatrix}1 & 0\\ b & 1\end{pmatrix}$ with b finite rank so that $D_Q(HGH^{-1})$ is invertible. Let $h=c\theta_Q \exp\frac{1}{2}R_Q(H)$ as above and write $g'_n=hg_nh^{-1}$. Then as above

$$\langle g'_n\rangle_Q=\langle g_n\rangle_{Q'} \qquad \text{where} \quad Q'=H^{-1}QH.$$

But $\langle g'_n\rangle_Q=c^{-1}\langle F_Q(h)\,F_Q(g_n)\,1_Q, 1_Q\rangle$ and since $F_Q(g_n)\,1_Q$ converges it follows that $\langle g'_n\rangle_Q=\langle g_n\rangle_{Q'}$ converges. This last limit must be non-zero since $\langle g'_n\rangle_Q^2$ converges to $\det D_Q(HGH^{-1})$. Q.E.D.

COROLLARY 3.6. *If g_n is a sequence in* $\mathrm{Spin}_0(W)$ *with $T(g_n)$ convergent in $SO_Q(W)$ and $\langle g_n\rangle_{Q'}$ converges to a non-zero value for some Q-finite involution Q' then $\langle g_n\rangle_{Q_1}$ converges for all Q-finite involutions Q_1.*

Proof. When $\lim_n D_{Q_1}(g_n)=D_1$ is invertible then Theorem 3.2 applies as in the proof of Proposition 3.5. When $\lim D_{Q_1}(g_n)$ is not invertible then $\lim_n\langle g_n\rangle_{Q_1}^2=\det D_1=0$ which implies $\lim_n\langle g_n\rangle_{Q_1}=0$. Q.E.D.

There is one further technical lemma that will be useful in the discussion of the group structure for $\mathrm{Spin}_Q(W)$:

LEMMA 3.7 *Suppose g_n and g'_n are two d_Q-Cauchy sequences in* $\mathrm{Spin}_0(W)$. *Suppose* $\lim_n T(g_n)=\lim_n T(g'_n)$ *and that there exists a Q-finite involution Q_1 such that* $\lim_n\langle g_n\rangle_{Q_1}=\lim_n\langle g'_n\rangle_{Q_1}\neq 0$. *Then*

$$\lim_n F_Q(g_n)\,1_Q=\lim_n F_Q(g'_n)\,1_Q.$$

Proof. The proof follows that of Proposition 3.5 so we will be sketchy.

Let $G = \lim_n T(g_n) = \lim_n T(g'_n)$ and suppose to begin with that $D_Q(G)$ is invertible. Then $\langle g_n \rangle_Q$ and $\langle g'_n \rangle_Q$ will both converge by Corollary 3.6 to something non-zero since $\det D_Q(G) \neq 0$. In fact using

$$\langle g_n \rangle_Q = \langle g_n \rangle_{Q_1} \exp\left\{ -\frac{1}{2} \int_\gamma \operatorname{Tr}((1+\lambda A_n)^{-1} A_n)\, d\lambda \right\},$$

where $A_n = (Q_1 - Q)\, R_Q(g_n)/2$ and the analogous result for g'_n with $A'_n = (Q_1 - Q)\, R_Q(g'_n)/2$, we see that $\langle g_n \rangle_Q$ and $\langle g'_n \rangle_Q$ converge to the same limit since A_n and A'_n converge to the same limit in trace norm. Now expand

$$\begin{aligned} &\|F_Q(g_n)\, 1_Q - F_Q(g'_n)\, 1_Q\|^2 \\ &\quad = \langle g_n^* g_n \rangle_Q + \langle (g'_n)^* g'_n \rangle_Q - \langle g_n^* g'_n \rangle_Q - \langle (g'_n)^* g_n \rangle_Q \end{aligned} \tag{3.5}$$

and use the formula

$$\langle h^* g \rangle_Q = \langle h^* \rangle_Q \langle g \rangle_Q \exp\left\{ \frac{1}{2} \int_\gamma \operatorname{Tr}[(1+\lambda Z_h^* Z_g)^{-1} Z_h^* Z_g]\, d\lambda \right\},$$

where $Z_g = B_Q(g)\, D_Q(g)^{-1}$. The products $Z_h^* Z_g$ where $h = g_n$ or g'_n and $g = g_n$ or g'_n all have the same limit in trace norm by hypothesis and we just showed that $\langle g_n \rangle_Q$ and $\langle g'_n \rangle_Q$ have the same limits as do $\langle g_n^* \rangle_Q = \overline{\langle g_n \rangle_Q}$ and $\langle (g'_n)^* \rangle_Q = \overline{\langle g'_n \rangle_Q}$. This shows that the inner products on the right hand side of (3.5) cancel one another in the limit $n \to \infty$.

If $D_Q(G)$ is not invertible then introduce H and h as in Proposition 3.5 so that $h \in \mathrm{Spin}_0(W)$, $T(h) = H$, and $D_Q(HGH^{-1})$ is invertible. Then $\langle h g_n h^{-1} \rangle_Q = \langle g_n \rangle_{H^{-1}QH}$ and $\langle h g'_n h^{-1} \rangle_Q = \langle g'_n \rangle_{H^{-1}QH}$. These converge to the same value since $\langle g_n \rangle_{Q_1}$ and $\langle g'_n \rangle_{Q_1}$ converge to the same value (the argument is as above). Thus by the preceding result $\lim_n F_Q(h g_n h^{-1})\, 1_Q = \lim_n F_Q(h g'_n h^{-1})\, 1_Q$. Since $F_Q(h^{-1})\, 1_Q = c^{-1} 1_Q$ this is easily seen to imply that $\lim_n F_Q(g_n)\, 1_Q = \lim_n F_Q(g'_n)\, 1_Q$. Q.E.D.

We are now prepared to discuss the group structure for $\mathrm{Spin}_Q(W)$. The elements of $\mathrm{Spin}_Q(W)$ we take to be equivalence classes of d_Q Cauchy sequences in $\mathrm{Spin}_0(W)$. Two d_Q Cauchy sequences g_n and g'_n in $\mathrm{Spin}_0(W)$ will be said to be equivalent if and only if $\lim_n T(g_n) = \lim_n T(g'_n)$ and $\lim_n F_Q(g_n)\, 1_Q = \lim_n F_Q(g'_n)\, 1_Q$. If g is such an equivalence class we write $T(g)$ for $\lim_n T(g_n)$ where g_n is any representative Cauchy sequence for g. If g_n and h_n are representative Cauchy sequences for g and h, elements of $\mathrm{Spin}_Q(W)$, then we define g^{-1} as the equivalence class of g_n^{-1} and gh as the equivalence classes of $g_n h_n$. For these definitions to make sense we must show that g_n^{-1} and $g_n h_n$ are indeed d_Q Cauchy sequences and that the definitions of g^{-1} and gh do not depend on the particular Cauchy sequences g_n and h_n chosen to represent g and h. We begin by dealing with adjoints and inverses.

PROPOSITION 3.8. *Suppose $g_n \in \mathrm{Spin}_0(W)$ is a d_Q Cauchy sequence, then g_n^* and g_n^{-1} are d_Q Cauchy sequences.*

Proof. First we will show that g_n^* is d_Q Cauchy. Let $G = \lim_n T(g_n)$ and suppose to begin with that $D_Q(G)$ is invertible. Then $\langle g_n^* \rangle_Q = \overline{\langle g_n \rangle_Q}$ converges to something non-zero. Proposition 3.5 applies and we see that g_n^* is d_Q-Cauchy. Now suppose $D_Q(G)$ is singular. Then we can find $h \in \mathrm{Spin}_0(W)$ so that $\langle h g_n h^{-1} \rangle_Q$ converges to something non-zero. This implies $\langle h^{*-1} g_n^* h^* \rangle_Q = \overline{\langle h g_n h^{-1} \rangle_Q}$ converges to something non-zero. But $\langle h^{*-1} g_n^* h^* \rangle_Q = \langle g_n^* \rangle_{H^*QH^{*-1}}$ where $H = T(h)$ and H^*QH^{*-1} is a Q-finite involution. Thus Proposition 3.5 applies and it follows that g_n^* is a d_Q-Cauchy sequence.

Now we show that g_n^{-1} is d_Q-Cauchy. Corollary 1.3 shows that for $G = \lim_n T(g_n)$ there exists a Q-finite involution Q' such that both $D_{Q'}(G)$ and $D_{Q'}(G^{-1})$ are invertible. For n sufficiently large Theorem 3.2 applies and we have

$$1 = \langle g_n^{-1} g_n \rangle_{Q'} = \langle g_n^{-1} \rangle_{Q'} \langle g_n \rangle_{Q'} \exp\left\{\frac{1}{2}\int_\gamma \mathrm{Tr}[(1+\lambda R_n)^{-1} R_n]\, d\lambda\right\},$$

where $R_n = D_{Q'}(g_n^{-1})^{-1} C_{Q'}(g_n^{-1}) B_{Q'}(g_n) D_{Q'}(g_n)^{-1}$. It is not hard to see that R_n converges in trace norm. Since $\langle g_n \rangle_{Q'}$ converges by Proposition 3.5 it follows that $\langle g_n^{-1} \rangle_{Q'}$ converges as well. Hence g_n^{-1} is d_Q-Cauchy by another application of Proposition 3.5. Q.E.D.

PROPOSITION 3.9. *If g_n and h_n are d_Q-Cauchy sequences in $\mathrm{Spin}_0(W)$ then so is $g_n h_n$.*

Proof. Using Proposition 3.5 it is enough to show that $\langle g_n h_n \rangle_{Q_1}$ converges to a non-zero value for some Q-finite involution Q_1. Let $G = \lim_n T(g_n)$ and $H = \lim_n T(h_n)$ and choose Q_1 so that $D_{Q_1}(GH)$ is invertible. It is then automatic that if $\langle g_n h_n \rangle_{Q_1}$ converges at all it will converge to something nonzero (i.e., with square equal to $\det D_{Q_1}(GH)$). For the Q-finite involution Q_1 there are algebraic vectors v_1 and $v_2 \in A_0(W_+)$ such that $\langle X \rangle_{Q_1} = \langle F_Q(X) v_1, v_2 \rangle$ for $X \in \mathscr{C}_0(W)$. Thus $\langle g_n h_n \rangle_{Q_1} = \langle F_Q(g_n) F_Q(h_n) v_1, v_2 \rangle = \langle F_Q(h_n) v_1, F_Q(g_n^*) v_2 \rangle$. By Proposition 3.8 we know that g_n^* is a d_Q-Cauchy sequence and this together with the fact that h_n is a d_Q-Cauchy sequence is easily seen to imply that $F_Q(h_n) v_1$ and $F_Q(g_n^*) v_2$ are strongly convergent. But this implies $\langle g_n h_n \rangle_{Q_1}$ is convergent. Q.E.D.

It remains to show that g^{-1} and gh do not depend on the choice of Cauchy sequences g_n and h_n. Suppose g'_n and h'_n are two other Cauchy sequences representing g and h. To show that $\lim_n g_n^{-1} = \lim_n g_n'^{-1}$ it is enough by Lemma 3.7 to exhibit a Q-finite involution Q' such that

$$\lim_n \langle g_n^{-1} \rangle_{Q'} = \lim_n \langle g_n'^{-1} \rangle_{Q'} \neq 0.$$

The Q-finite involution Q', used in Proposition 3.8, will do the job since $\lim_n \langle g_n^{-1} \rangle_{Q'}$ is expressed in terms of $\lim_n \langle g_n \rangle_{Q'}$ and the matrix elements of $\lim_n T(g_n)$. But $\lim_n \langle g_n \rangle_{Q'} = \lim_n \langle g'_n \rangle_{Q'}$ and $\lim_n T(g_n) = \lim_n T(g'_n)$ since g_n and g'_n both represent g. Thus $\lim_n \langle g_n^{-1} \rangle_{Q'} = \lim_n \langle g_n'^{-1} \rangle_{Q'}$.

Similar considerations apply to $g_n h_n$ and $g'_n h'_n$. In the proof of Proposition 3.9 we found $\langle g_n h_n \rangle_{Q_1} = \langle F_Q(h_n) v_1, F_Q(g_n^*) v_2 \rangle$ (and the analogue with g_n and h_n replaced with g'_n and h'_n) where v_1 and v_2 are algebraic vectors in $A_0(W_+)$. Each of the vectors v_j $(j = 1, 2)$ is a finite sum of vectors of the form $F_Q(w_1 w_2 \cdots w_k) 1_Q$ where $w_1, ..., w_k$ are elements of W regarded as elements of $\mathscr{C}_0(W)$. But $F_Q(h_n) F_Q(w_1 \cdots w_k) 1_Q = F_Q(T(h_n) w_1) F_Q(T(h_n) w_2) \cdots F_Q(T(h_n) w_k) F_Q(h_n) 1_Q$. Since $x \to F_Q(x)$ is continuous from the Hilbert space topology for x to the uniform operator topology for $F_Q(x)$ it follows that the limit as $n \to \infty$ of this last vector is

$$F_Q(T(h) w_1) \cdots F_Q(T(h) w_n) F_Q(h) 1_Q,$$

where we have written $T(h) = \lim_n T(h_n)$ $F_Q(h) 1_Q = \lim_n F_Q(h_n) 1_Q$. The limit is clearly independent of the choice of Cauchy sequence representing h and we have $\lim_{n \to \infty} F_Q(h_n) v_1 = \lim_n F_Q(h'_n) v_1$. A similar result for g_n^* and $(g'_n)^*$ finishes the demonstration that $\lim_n \langle g_n h_n \rangle = \lim_n \langle g'_n h'_n \rangle_{Q_1} \neq 0$. We have shown that multiplication in $\mathrm{Spin}_Q(W)$ is well defined.

We have almost finished the proof of Theorem 3.4. It remains to be shown that the homomorphism $T: \mathrm{Spin}_0(W) \to SO_0(W)$ extends to a surjective homomorphism $T: \mathrm{Spin}_Q(W) \to SO_Q(W)$. First we prove that $SO_0(W)$ is dense in $SO_Q(W)$. Suppose $G \in SO_Q(W)$. Then Lemma 1.2 guarantees the existence of a Q-finite involution Q' such that $D_{Q'}(G)$ is invertible. Let $R' = (G-1)(Q'_- G + Q'_+)^{-1}$. Then $G = (1 - R'Q'_-)^{-1} (1 + R'Q'_+)$ and $(R')^\tau = -R'$. Using the fact that $(Q'_\pm)^\tau = Q'_\pm$ it is easy to check that if R is any skew symmetric map on W $(R^\tau = -R)$ then $(1 - RQ'_-)^{-1} (1 + RQ'_+)$ will be in $SO(W)$ provided that $(1 - RQ'_-)$ is invertible. Now let P_n denote a sequence of finite rank orthogonal projections on W converging strongly to the identity with the further property that the range of each P_n is a Q' and P invariant subspace of W. Then $P_n^* = P_n$ and $P_n^\tau = P_n$. Define $R'_n = P_n R' P_n$. Then $(R'_n)^\tau = -R'_n$. Furthermore since R' is compact R'_n converges uniformly to R'. Since $1 - R'Q'_-$ is invertible it follows that $1 - R'_n Q'_-$ will be invertible for all sufficiently large n, and we define $G_n = (1 - R'_n Q'_-)^{-1} (1 + R'_n Q'_+)$ (obviously $G_n \in SO_0(W)$ since P_n is finite rank). Since each P_n commutes with Q' the diagonal matrix elements of R'_n relative to the $Q'_+ W + Q'_- W$ splitting of W are obtained from the diagonal matrix of R' by pre- and postmultiplying by P_n. The diagonal elements of R' are trace class and since the product of a strongly convergent sequence with one that converges in trace norm also converges in trace norm it follows that the diagonal elements of R'_n converge in trace norm to the diagonal elements of R'. For precisely analogous reasons the off diagonal matrix elements of R'_n converge in Schmidt norm to the off diagonal elements of R'. Without difficulty this is seen to imply that G_n converges to G in trace norm on the diagonal and in Schmidt norm on the off diagonal relative to the $Q'_+ W + Q'_- W$ splitting. But $Q - Q'$ is finite rank so the same thing is true for the matrix elements in the $Q_+ W + Q_- W$ splitting of W. Thus G_n converges to G in $SO_Q(W)$. It remains to show that G_n is covered by a d_Q Cauchy sequence in $\mathrm{Spin}_0(W)$. Choose $g_n \in \mathrm{Spin}_0(W)$ such that $T(g_n) = G_n$. Then $\lim_n \langle g_n \rangle^2_{Q'} = \det D_{Q'}(G) \neq 0$. By inserting $\pm$ signs in g_n we may thus ensure that $\lim_n \langle g_n \rangle_{Q'}$ converges to something non-zero. Proposition 3.5 implies that the resulting sequence is d_Q-Cauchy. Since $T(\pm g_n) = T(g_n) = G_n$ we have finished the proof of Theorem 3.4.

Let Θ denote the orbit of 1_Q under the action of $\mathrm{Spin}_Q(W)$. That is $\Theta \subseteq A(W_+)$ consists of those vectors $v \in A(W_+)$ such that $v = \lim_n F_\Omega(g_n) 1_Q$ for some d_Q Cauchy sequence $g_n \in \mathrm{Spin}_0(W)$. Let $\mathscr{D}$ denote the dense linear domain contained in $A(W_+)$ obtained from Θ by the action of the *algebraic* Clifford algebra $\mathscr{C}_0(W)$ in the Q-Fock representation. That is $\mathscr{D} = \mathscr{C}_0(W)\Theta$. Our next object in this section is:

THEOREM 3.10. *There is a strongly continuous representation Γ_Q of* $\mathrm{Spin}_Q(W)$ *on $\mathscr{D}$ such that*

$$\Gamma_Q(g) F_Q(w) = F_Q(T(g)w) \Gamma_Q(g)$$

for $w \in W \subseteq \mathscr{C}_0(W)$. The equality is understood as an equality between densely defined operators on $\mathscr{D}$.

Proof. Suppose $g \in \mathrm{Spin}_Q(W)$ and that g_n is a representative Cauchy sequence for g. It is natural to define $\Gamma_Q(g) 1_Q = \lim_n F_Q(g_n) 1_Q$. If $v \in \mathscr{D}$ then we would like to define $\Gamma_Q(g) v = \lim_n F_Q(g_n) v$. We must show that this definition makes sense. The vector v is a finite sum of vectors of the form $F_Q(w_1 \cdots w_k) \Gamma_Q(h) 1_Q$ where $w_j \in W \subseteq \mathscr{C}_0(W)$ $(j = 1, ..., k)$ and $h \in \mathrm{Spin}_Q(W)$. But $F_Q(g_n) F_Q(w_1 \cdots w_k) \Gamma_Q(h) 1_Q = F_Q(T(g_n) w_1) \cdots F_Q(T(g_n) w_k) F_Q(g_n) \Gamma_Q(h) 1_Q$. Since the sequences $F_Q(T(g_n) w_j)$ converge in uniform norm to $F_Q(T(g) w_j)$ $(j = 1, ..., k)$ to show that $F_Q(g_n)v$ converges it is enough to prove that $F_Q(g_n) \Gamma_Q(h) 1_Q$ converges. Let h_j be a representative Cauchy sequence for h and choose j_n large enough so that

$F_Q(g_n)\, F_Q(h_{j_n})\, 1_Q$ differs from $F_Q(g_n)\, \Gamma_Q(h)\, 1_Q$ by less than $1/n$ (in norm (this is possible since $F_Q(g_n)$ is a bounded operator on $A(W_+)$). But now $F_Q(g_n)\, F_Q(h_{j_n})\, 1_Q = F_Q(g_n h_{j_n})\, 1_Q$. By Proposition 3.9, $g_n h_{j_n}$ is a Cauchy representative for $g \cdot h$ and we find

$$\lim_n F_Q(g_n) \Gamma_Q(h)\, 1_Q = \lim_n F_Q(g_n h_{j_n})\, 1_Q = \Gamma_Q(gh)\, 1_Q.$$

Observe that this proof also shows that the $\lim_n F_Q(g_n)v$ does not depend on which representative Cauchy sequence g_n is chosen for g. For essentially the same reason it is clear that $\Gamma_Q(g)\, \Gamma_Q(h)v = \Gamma_Q(gh)v$ and that $\Gamma_Q(g) F_Q(w)v = F_Q(T(g)w)\, \Gamma_Q(g)v$. It remains to check that Γ_Q is strongly continuous. Suppose g_n is a sequence in $\mathrm{Spin}_Q(W)$ converging to g in $\mathrm{Spin}_Q(W)$. Let $v \in \mathscr{D}$. For each n let g_{nj_n} $(j = 1, 2, \ldots)$ denote a sequence in $\mathrm{Spin}_0(W)$ converging to g_n in $\mathrm{Spin}_Q(W)$ as $j \to \infty$. Choose j_n large enough so that g_{nj_n} converges to g as $n \to \infty$ in $\mathrm{Spin}_Q(W)$ and so that $\Gamma_Q(g_n)v$ differs from $\Gamma_Q(g_{nj_n})v$ by less than $1/n$ in norm. It is then clear that $\lim_n \Gamma_Q(g_n)v = \lim_n \Gamma_Q(g_{nj_n})v = \Gamma_Q(g)v$. Q.E.D.

4. The Groups of $\widehat{\mathrm{Spin}}_Q(W)$ and $\widehat{GL}_Q(H)$

In this section we will construct the group $\widehat{\mathrm{Spin}}_Q(W)$ described in the Introduction. Let $GL(W_+)$ denote the general linear group of bounded invertible transformations on W_+. Our first goal is to show that $GL(W_+)$ acts by automorphisms on $\mathrm{Spin}_Q(W)$. Let $G \in GL(W_+)$. Then $G \oplus G^{-\tau}$ is a complex orthogonal map on W which extends to an (algebra) automorphism of $\mathscr{C}_0(W)$ which we denote by $\alpha(G)$. Thus if $w_j \in W$ $(j = 1, \ldots, k)$ then

$$\alpha(G)\, w_1 \cdots w_k = (G \oplus G^{-\tau})\, w_1 \cdots (G \oplus G^{-\tau})\, w_k.$$

Since every element in $\mathrm{Spin}_0(W)$ is an even product of elements from W and since $\alpha(G)$ clearly leaves the spinor norm invariant, it follows that $\alpha(G)$ restricts to an automorphism of $\mathrm{Spin}_0(W)$. Suppose $w \in W$ and $(w, w) \neq 0$. Then for $x \in W$ we have

$$\begin{aligned} T(\alpha(G)w)x &= T(G \oplus G^{-\tau} w)x \\ &= x - 2\frac{(G \oplus G^{-\tau} w, x)}{(w, w)} (G \oplus G^{-\tau})w \\ &= (G \oplus G^{-\tau})\, T(w) (G \oplus G^{-\tau})^{-1}\, x, \end{aligned}$$

where we used the fact that $G \oplus G^{-\tau}$ is complex orthogonal. Again, every element in $\mathrm{Spin}_0(W)$ is a product of $w_j \in W$ with $(w_j, w_j) \neq 0$ so it follows that for $g \in \mathrm{Spin}_0(W)$ we have

$$T(\alpha(G)\, g) = (G \oplus G^{-\tau})\, T(g)(G \oplus G^{-\tau})^{-1}. \tag{4.1}$$

Next we wish to show that $\alpha(G)$ extends to an automorphism of $\mathrm{Spin}_Q(W)$. We begin by showing that if $g_n \in \mathrm{Spin}_0(W)$ is a d_Q-Cauchy sequence then $\alpha(G)\, g_n$ is also a d_Q-Cauchy sequence for $G \in GL(W_+)$. Since $T(\alpha(G)\, g_n) = (G \oplus G^{-\tau})\, T(g_n)(G \oplus G^{-\tau})^{-1}$ and $T(g_n)$ converges in $SO_0(W)$ it is trivial to check that $T(\alpha(G)\, g_n)$ converges in $SO_Q(W)$. Thus we need only show that $F_Q(\alpha(G)\, g_n)\, 1_Q$ converges in $A(W_+)$. Let $g = \lim_n g_n$, $T(g) = \lim_n T(g_n) = \left[\begin{smallmatrix} A(g) & B(g) \\ C(g) & D(g) \end{smallmatrix}\right]$. Assume to begin with that $D(g)$ is invertible. Then $\langle g_n \rangle_Q$ converges to a non-zero value since its square converges to $\det D(g)$. But because $G \oplus G^{-\tau}$ respects the $W_+ \oplus W_-$ splitting of W it is easily verified that $\langle \alpha(G) X \rangle_Q = \langle X \rangle_Q$ for $X \in \mathscr{C}_0(W)$. Thus $\lim_n \langle \alpha(G)\, g_n \rangle_Q = \lim_n \langle g_n \rangle_Q \neq 0$. By Proposition 3.5 it follows that $\alpha(G)\, g_n$ is d_Q-Cauchy. If $D(g$ is not invertible then we can find $h \in \mathrm{Spin}_0(W)$ with $T(h) = H = \left[\begin{smallmatrix} 1 & 0 \\ b & 1 \end{smallmatrix}\right]$, b finite rank, such that $HT(g)\, H^{-1}$ does have an invertible D_Q matrix element. Let $g'_n = h g_n h^{-1}$. Since $F_Q(h^{-1})\, 1_Q = c 1_Q$ for some constant c it is clear that g'_n is a d_Q-Cauchy sequence in $\mathrm{Spin}_0(W)$. Since $\langle g'_n \rangle_Q$ converges to a non-zero value it follows that $\alpha(G)\, g'_n$ is d_Q-Cauchy by the preceding argument. But $\alpha(G)\, g_n = \alpha(G)(h^{-1} g'_n h) = [\alpha(G)\, h^{-1}][\alpha(G)\, g'_n][\alpha(G) h]$. Making use of $F_Q(\alpha(G) h)\, 1_Q = F_Q(h)\, 1_Q = c^{-1} 1_Q$, we find $F_Q(\alpha(G)\, g_n)\, 1_Q = c^{-1} F_Q(\alpha(G)\, h^{-1})\, F_Q(\alpha(G)\, g'_n)\, 1_Q$ which converges in $A(W_+)$ since $F_Q(\alpha(G) h^{-1})$ is bounded and $\alpha(G)\, g'_n$ is d_Q-Cauchy. If $g \in \mathrm{Spin}_Q(W)$ and $g_n \in \mathrm{Spin}_0(W)$ converges to g in $\mathrm{Spin}_Q(W)$ we would like to define

$$\alpha(G)\, g = \lim_n \alpha(G)\, g_n.$$

To see that this makes sense we need only check that it does not depend on the choice of Cauchy sequence g_n which approximates g. If $D(g)$ is invertible then this is implied by Lemma 3.7. If $D(g)$ is not invertible the modifications needed in the argument are the usual ones and are left to the reader. We also leave to the reader the verification of the fact that $G \to \alpha(G)$ is a homomorphism and that $\alpha(G)\colon \mathrm{Spin}_Q(W) \to \mathrm{Spin}_Q(W)$ is a *continuous* automorphism. We summarize these developments in the following proposition:

PROPOSITION 4.1. *The automorphism of $\mathscr{C}_0(W)$ induced by the complex orthogonal $G \oplus G^{-\tau}$ for $G \in GL(W_+)$ restricts to an automorphism $\alpha(G)$ of* $\mathrm{Spin}_0(W)$. *The automorphism $\alpha(G)$ extends uniquely to a continuous* *automorphism of* $\mathrm{Spin}_Q(W)$. *The map* $G \to \alpha(G)$ *is a homomorphism and* $T(\alpha(G)\, g) = (G \oplus G^{-\tau}) T(g)(G \oplus G^{-\tau})^{-1}$.

Next we define a representation Γ of $GL(W_+)$ on the dense invariant domain $\mathscr{D} \subseteq A(W_+)$ which is described at the end of Section 3. For $G \in GL(W_+)$ we define

$$\Gamma(G) = \sum_{n=0}^{\infty} G^{(n)},$$

where $G^{(0)} = Id$ on $\mathbb{C}$ and $G^{(n)} = G \otimes \cdots \otimes G$ (n factors) acts on $\Lambda^n(W_+)$. In general $\Gamma(G)$ will be unbounded on all of $A(W_+)$ but it is certainly well defined on the algebraic tensors $\mathscr{C}_0(W)\, 1_Q$. If $X \in \mathscr{C}_0(W)$ then it is well known (and easily checked) that

$$\Gamma(G)\, F_Q(X) = F_Q(\alpha(G) X)\, \Gamma(G), \tag{4.2}$$

where the equality is understood between operators acting on $\mathscr{C}_0(W)\, 1_Q$. We wish to extend $\Gamma(G)$ to a transformation on $\mathscr{D}$. Suppose $w_j \in W$ ($j = 1, ..., k$) and g_n is a d_Q-Cauchy sequence in $\mathrm{Spin}_0(W)$ converging to g in $\mathrm{Spin}_Q(W)$. It is enough to prove that $n \to \Gamma(G)\, F_Q(w_1 \cdots w_k)\, F_Q(g_n)\, 1_Q$ converges to an element in $\mathscr{D}$. But

$$\begin{aligned} &\Gamma(G)\, F_Q(w_1 \cdots w_k)\, F_Q(g_n)\, 1_Q \\ &\quad = F_Q(\alpha(G)\, w_1 \cdots w_k)\, F_Q(\alpha(G)\, g_n)\, 1_Q, \end{aligned}$$

where $\alpha(G)\, w_1 \cdots w_k = (G \oplus G^{-\tau})\, w_1 \cdots (G \oplus G^{-\tau})\, w_k$. We already showed that $\alpha(G)\, g_n$ is d_Q-Cauchy and since each $F_Q(G \oplus G^{-\tau} w_j)$ ($j = 1, ..., k$) is a bounded operator it follows that the sequence in question does converge to an element in $\mathscr{D}$. It is clear that (4.2) remains true as an equality on $\mathscr{D}$ and by passing to limits in the X variable in (4.2) one also has

$$\Gamma(G)\, \Gamma_Q(g) = \Gamma_Q(\alpha(G)\, g)\, \Gamma(G) \tag{4.3}$$

as an equality on $\mathscr{D}$ for $g \in \mathrm{Spin}_Q(W)$.

Let $\mathrm{Spin}_Q(W) \times_+ GL(W_+)$ denote the semi-direct product with the composition $(g_1 \times G_1)(g_2 \times G_2) = g_1 \alpha(G_1)\, g_2 \times G_1 G_2$. Then (4.3) implies that $\Gamma_Q \times \Gamma(g \times G) \overset{\text{def}}{=} \Gamma_Q(g)\, \Gamma(G)$ is a homomorphism from $\mathrm{Spin}_Q(W) \times_\alpha GL(W_+)$ into $L(\mathscr{D})$, the linear maps on $\mathscr{D}$ which leave $\mathscr{D}$ invariant. Let ker denote the kernel of this homomorphism and write

$$\widehat{\mathrm{Spin}}_Q(W) = \mathrm{Spin}_Q(W) \times_\alpha GL(W_+)/\mathrm{ker}.$$

Let T denote the map from $\widehat{\mathrm{Spin}}_Q(W)$ to $SO_{\mathrm{res}}(W)$ which sends the coset $g \times G(\mathrm{ker})$ into $T(g)[G \oplus G^{-\tau}]$. It is simple to check that this is well defined and a homomorphism. In the remarks preceding Lemma 1.1 we showed that $SO_{\text{res}}(W)$ is the product of $SO_Q(W)$ and $GL(W_+)$ so $T\colon \widehat{\text{Spin}}_Q(W) \to SO_{\text{res}}(W)$ is surjective. Let S denote the subgroup of $\text{Spin}_Q(W)$ consisting of those $g \in \text{Spin}_Q(W)$ such that $T(g) = G \oplus G^{-\tau}$ respects the $W_+ \oplus W_-$ splitting of W. Observe that for such a g the map G is a trace class perturbation of the identity on W_+. Let $\hat{S}$ denote the subgroup of $\widehat{\text{Spin}}_Q(W)$ which consists of cosets of the form $g \times G^{-1}(\ker)$ where $g \in S$ and $T(g) = G \oplus G^{-1}$. Then it is easily seen that $\hat{S}$ is the kernel of the homomorphism T. Furthermore, $\hat{S}$ maps into multiples of the identity under the homomorphism $\Gamma_Q \times \Gamma$. Indeed $\Gamma_Q(g)\Gamma(G^{-1}) = \langle g \rangle_Q Id$ for $g \times G^{-1}(\ker) \in \hat{S}$. Since $\langle g \rangle_Q^2 = \det D(g)$ it is clear that every non-zero multiple of the identity occurs and hence that $\hat{S} \simeq \mathbb{C}^*$. We have then the exact sequence

$$\mathbb{C}^* \longrightarrow \widehat{\text{Spin}}_Q(W) \xrightarrow{T} SO_{\text{res}}(W) \longrightarrow 0,$$

where $\mathbb{C}^*$ is identified with $\hat{S}$.

Let $\hat{\Gamma}_Q$ denote the homomorphism from $\widehat{\text{Spin}}_Q(W)$ into $L(\mathscr{D})$ induced by the homomorphism $\Gamma_Q \times \Gamma$ which maps $\text{Spin}_Q(W) \times_\alpha GL(W_+)$ into $L(\mathscr{D})$. Then if we compare Theorem 3.10 with (4.2) and the definition of $T\colon \widehat{\text{Spin}}_Q(W) \to SO_{\text{res}}(W)$, then

$$\hat{\Gamma}_Q(g)\, F_Q(x)\, \hat{\Gamma}_Q(g)^{-1} = F_Q(T(g)x) \qquad \text{for} \quad x \in W,\ g \in \widehat{\text{Spin}}_Q(W),$$

where the equality is equality in $L(\mathscr{D})$.

As mentioned in the Introduction the construction of $\widehat{\text{Spin}}_Q(W)$ and the representation $\hat{\Gamma}_Q$ implementing the automorphisms of $\mathscr{C}_0(W)$ induced by elements of $SO_{\text{res}}(W)$ are the main results of this paper.

Theorem 1.2 in [15] shows that every complex orthogonal in $O_{\text{res}}(W)$ differs from an element of $SO_{\text{res}}(W)$ by the action of a single complex orthogonal reflection. The proof given there is for finite dimensional complex orthogonals but only requires that the "A" and "D" matrix elements of a complex orthogonal are Fredholm with index 0. This is true for the group $O_{\text{res}}(W)$ and as mentioned in the Introduction by combining the action of $\widehat{\text{Spin}}_Q(W)$ and $\mathscr{C}(W)$ on $L(\mathscr{D})$ one may cover the automorhisms of $\mathscr{C}(W)$ induced by elements of $O_{\text{res}}(W)$ [see [20] for more details].

Next we would like to make the connection with the group $\widehat{GL}(H)$ constructed by G. Segal and G. Wilson [35]. Let Q_0 denote a distinguished symmetric involution on W which anti-commutes with P. Let H denote the $+1$ eigenspace for Q_0. Since P anti-commutes with Q_0 and is conjugate linear we may use P to identify the -1 eigenspace for Q_0 with the conjugate Hilbert space $\bar{H}$. The isotropic decomposition of W associated with Q_0 may thus be written $W = H \oplus \bar{H}$. The conjugation P has matrix $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ and the complex structure has matrix $\left[\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right]$ relative to this splitting of W.

Let $GL_0(H)$ denote the group of invertible transformations on H which are finite rank perturbations of the identity. If $G \in GL_0(H)$ observe that $G \oplus G^{*-1} \in SO_0(W)$. Consider the subgroup of $G_0(W)$ consisting of those g in $G_0(W)$ with $T(g) = G \oplus G^{*-1}$ and $G \in GL_0(H)$. In Section 2 we saw that there is a homomorphism $g \to \langle g \rangle_0$ defined from this subgroup of $G_0(W)$ into $\mathbb{C}^*$. This homomorphism is "linear" in g so that for each $G \in GL_0(H)$ there is a unique $\gamma(G) \in G_0(W)$ such that

$$\langle \gamma(G) \rangle_0 = 1$$

and

$$T(\gamma(G)) = G \oplus G^{*-1}.$$

Since $g \to \langle g \rangle_0$ is a homomorphism it is clear that $G \to \gamma(G) \in G_0(W)$ is also a homomorphism. The map γ does not map into $\mathrm{Spin}_0(W)$. Instead by Theorem 2.1 we have

$$1 = \langle \gamma(G) \rangle_0^2 = nr(\gamma(G)) \det(G^{*-1} \mid \bar{H}),$$

where $(G^{*-1} \mid \bar{H})$ is the map G^{*-1} acting on $\bar{H}$.

We have been explicit about the map G^{*-1} acting on $\bar{H}$ since $\det(G \mid \bar{H}) = \overline{\det(G \mid H)}$. Thus $\det(G^{*-1} \mid \bar{H}) = \det(G^* \mid \bar{H})^{-1} = \det(G \mid H)^{-1}$. It follows then that $nr(\gamma(G)) = \det G$ (when G is now understood to be acting on H). Let $SL_0(H)$ denote the subgroup of $GL_0(H)$ with determinant equal to 1. It is clear that γ restricts to a homomorphism $\gamma\colon SL_0(H) \to \mathrm{Spin}_0(W)$. We are interested in continuous extensions of this homomorphism into $\mathrm{Spin}_Q(W)$ for special choices of Q which we now describe. Let Q_H be a symmetric involution on H. If we define $Q = Q_H \oplus (-Q_H)$ on $H \oplus \bar{H} = W$ then Q is a symmetric involution on W which anti-commutes with P. Let $Q_H^{\pm} = (1 \pm Q_H)/2$ and write $H = H_+ \oplus H_-$ where $H_{\pm} = Q_H^{\pm} H$. If G is a linear map on H then we write $G = \left[\begin{smallmatrix} a(G) & b(G) \\ c(G) & d(G) \end{smallmatrix}\right]$ for the matrix of G relative to the orthogonal direct sum decomposition $H_+ \oplus H_-$ of H. Let $SL_Q(H)$ denote the sequential closure of $SL_0(H)$ in the topology of trace norm convergence for the diagonal matrix elements "a" and "d" and convergence in Schmidt norm for the off diagonal matrix elements "b" and "c." We have:

PROPOSITION 4.2. *The homomorphism $\gamma\colon SL_0(H) \to \mathrm{Spin}_0(W)$ extends continuously to a homomorphism $\gamma\colon SL_Q(H) \to \mathrm{Spin}_Q(W)$.*

Proof. Suppose $G \in SL_Q(H)$ and that G_n is a sequence in $GL_0(H)$ such that $a(G_n)$ and $d(G_n)$ converge in trace norm to $a(G)$ and $d(G)$, respectively, and such that $b(G_n)$ and $c(G_n)$ converge in Schmidt norm to $b(G)$ and $c(G)$, respectively. We want to show that $\gamma(G_n)$ is d_Q convergent. It is simple to see that $G_n \oplus G_n^{*-1}$ converges in $SO_Q(W)$. It remains to show that $F_Q(\gamma(G_n))\, 1_Q$ converges. We calculate

$$\begin{aligned}
&\|F_Q(\gamma(G_n))\, 1_Q - F_Q(\gamma(G_m))\, 1_Q\|^2 \\
&\quad = \langle \gamma(G_n)^* \gamma(G_n)\rangle_Q + \langle \gamma(G_m)^* \gamma(G_m)\rangle_Q \\
&\qquad - \langle \gamma(G_n)^* \gamma(G_m)\rangle_Q - \langle \gamma(G_m)^* \gamma(G_n)\rangle_Q.
\end{aligned} \tag{4.4}$$

But we have

$$\langle \gamma(G_\alpha)^* \gamma(G_\beta)\rangle_Q = \langle \gamma(G_\alpha^* G_\beta)\rangle_Q = \det[d(G_\alpha^* G_\beta)]$$

using Theorem 2.2 and the fact that $\langle \gamma(G_\alpha^* G_\beta)\rangle_0 = 1$. Since $d(G_\alpha^* G_\beta) = d(G_\alpha)^* d(G_\beta) + b(G_\alpha)^* b(G_\beta)$ it follows that

$$\langle \gamma(G_\alpha)^* \gamma(G_\beta)\rangle_Q = \det(d(G_\alpha)^* d(G_\beta) + b(G_\alpha)^* b(G_\beta)).$$

For large α, β the argument of this last determinant gets arbitrarily close to $d(G)^* d(G) + b(G)^* b(G)$ in trace norm. Thus the terms on the right hand side of (4.4) cancel in the limit of large m and n. We have shown that $\gamma(G_n)$ is d_Q Cauchy and this finishes the proof of the proposition. Q.E.D.

We may extend the homomorphism in Proposition 4.2. Let $GL(H_+ \oplus H_-)$ denote the group of invertible transformations on H which respect the decomposition $H = H_+ \oplus H_-$. The elements G in $GL(H_+ \oplus H_-)$ have diagonal matrices $a(G) \oplus d(G)$ and there is a homomorphism from $GL(H_+ \oplus H_-)$ into $GL(W_+)$ (where $W_+ = H_+ \oplus \bar{H}_-$) given by $a(G) \oplus d(G) \to a(G) \oplus d(G)^{*-1}$ (note that $a(G) \oplus d(G)^{*-1}$ is just the restriction of the complex orthogonal $G \oplus G^{*-1}$ to W_+). For $G \in GL(H_+ \oplus H_-)$ let $\alpha_0(G)$ denote the automorphism of $\mathscr{C}_0(W)$ induced by the action of the complex orthogonal $G \oplus G^{*-1}$ on W. Since $G \oplus G^{*-1}$ also respects the $W_+ \oplus W_-$ decomposition of W, Proposition 4.1 implies that $\alpha_0(G)$ extends to an automorphism of $\mathrm{Spin}_Q(W)$. Since $G \oplus G^{*-1}$ commutes with Q_0 it follows that $\langle \alpha_0(G) X\rangle_0 = \langle X\rangle_0$ for $X \in \mathscr{C}_0(W)$. Thus $\alpha_0(G)$ leaves invariant the subgroup $\gamma(SL_Q(H)) \subseteq \mathrm{Spin}_Q(W)$ and one has

$$\alpha_0(G_1)\, \gamma(G_2) = \gamma(G_1 G_2 G_1^{-1}) \qquad \text{for} \quad G_1 \in GL(H_+ \oplus H_-)$$

and $G_2 \in SL_Q(H)$ since the determinant is a similarity invariant. Let $SL_Q(H) \times_0 GL(H_+ \oplus H_-)$ denote the semi-direct product with composition rule $G_1 \times G_2 \cdot G_1' \times G_2' = G_1 G_2 G_1' G_2^{-1} \times G_2 G_2'$. We may assemble the observations of this paragraph by concluding that there is a homomorphism $\hat{\gamma}$ from $SL_Q(H) \times_0 GL(H_+ \oplus H_-)$ into $\widehat{\mathrm{Spin}}_Q(W)$ given by $\hat{\gamma}(G \times (a \oplus d)) = \gamma(G) \times (a \oplus d^{*-1})\,\mathrm{ker}$. Write $\widehat{GL}_Q(H) = SL_Q(H) \times_0$

$GL(H_+ \oplus H_-)/\ker(\hat{\gamma})$. Then $\hat{\gamma}$ induces a homomorphism (which we continue to denote by $\hat{\gamma}$)

$$\hat{\gamma}\colon \hat{G}L_Q(H) \to \widehat{\mathrm{Spin}}_Q(W). \tag{4.5}$$

While it is not yet evident, the group $\hat{G}L_Q(H)$ is the group constructed by G. Segal and G. Wilson [35]. We will now explain this connection. The homomorphism $T \circ \hat{\gamma}$ obviously induces a homomorphism T_0 obtained by restricting $T \circ \hat{\gamma}(g)$ to the invariant subspace H for each $g \in \hat{G}L_Q(H)$. Our next goal is to prove that

$$\mathbb{C}^* \longrightarrow \hat{G}L_Q(H) \xrightarrow{T_0} GL_{\mathrm{res}}(H) \longrightarrow 0 \tag{4.6}$$

is an exact sequence. Here $GL_{\mathrm{res}}(H)$ denotes the group of invertible maps on H which have Schmidt class commutators with $Q_H = (G \mid H)$ and which have "a" and "d" matrix elements which are Fredholm maps with index 0. The only implication of (4.6) which is not obvious is that T_0 is a surjection on $GL_{\mathrm{res}}(H)$. This is a consequence of the following lemma (which was pointed out to us by D. Pickrell).

LEMMA 4.3. *Let $GL_Q(H)$ denote the group of invertible operators on H with matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ relative to the $H_+ \oplus H_-$ decomposition of H such that a and d are trace class perturbations of the identity on H_+ and H_- and b and c are Schmidt class operators. Then* $\det(\cdot)$ *has a continuous extension from $GL_0(H)$ to $GL_Q(H)$.*

Proof. Let $G \in GL_Q(H)$. Since $GL_0(H)$ is dense in $GL_Q(H)$ in the topology of trace norm convergence on the diagonal and Schmidt norm convergence on the off diagonal we can find a sequence $G_n \in GL_0(H)$ which converges to G in this sense. We write $G_n = \left(\begin{smallmatrix} 1+\alpha_n & \beta_n \\ \gamma_n & 1+\delta_n \end{smallmatrix}\right)$. Let $G_n(\lambda) = \left(\begin{smallmatrix} 1+\lambda\alpha_n & \lambda\beta_n \\ \lambda\gamma_n & 1+\lambda\delta_n \end{smallmatrix}\right)$ and $G(\lambda) = \left[\begin{smallmatrix} 1+\lambda\alpha & \lambda\beta \\ \lambda\gamma & 1+\lambda\delta \end{smallmatrix}\right]$ and join 0 to 1 in $\mathbb{C}$ by a smooth simple curve γ which avoids the values of λ for which $G(\lambda)$ is singular (there are only finitely many such λ in any compact region for $\mathbb{C}$). For sufficiently large n this path will also avoid the values of λ for which $G_n(\lambda)$ is singular. Thus

$$\begin{aligned} \log\det G_n &= \int_\gamma \frac{d}{d\lambda} \log\det G_n(\lambda)\, d\lambda \\ &= \int_\gamma \mathrm{Tr}(G_n(\lambda)^{-1}\, dG_n(\lambda)/d\lambda)\, d\lambda. \end{aligned}$$

Let $G_n(\lambda)^{-1} = \left(\begin{smallmatrix} a_n(\lambda) & b_n(\lambda) \\ c_n(\lambda) & d_n(\lambda) \end{smallmatrix}\right)$ and $G(\lambda)^{-1} = \left(\begin{smallmatrix} a(\lambda) & b(\lambda) \\ c(\lambda) & d(\lambda) \end{smallmatrix}\right)$. Then for λ on the path, γ, $a_n(\lambda)$ and $d_n(\lambda)$ converge in trace norm to $a(\lambda)$ and $d(\lambda)$ and $b_n(\lambda)$ and

$c_n(\lambda)$ converge in Schmidt norm to $b(\lambda)$ and $c(\lambda)$. The convergence is uniform in the parameter λ for λ on γ. But

$$\begin{aligned}
&\operatorname{Tr}(G_n(\lambda)^{-1}\, dG_n(\lambda)/d\lambda) \\
&\quad = \operatorname{Tr}\begin{bmatrix} a_n(\lambda)\,\alpha_n + b_n(\lambda)\,\gamma_n & a_n(\lambda)\,\beta_n + b_n(\lambda)\,\delta_n \\ c_n(\lambda)\,\alpha_n + d_n(\lambda)\,\gamma_n & c_n(\lambda)\,\beta_n + d_n(\lambda)\,\delta_n \end{bmatrix} \\
&\quad = \operatorname{Tr}(a_n(\lambda)\,\alpha_n + b_n(\lambda)\,\gamma_n) + \operatorname{Tr}(c_n(\lambda)\,\beta_n + d_n(\lambda)\,\delta_n).
\end{aligned}$$

Since the product of two Schmidt norm convergent sequences also converges in trace norm it follows that $\lim_n \operatorname{Tr}(G_n(\lambda)^{-1}\, dG_n(\lambda)/d\lambda) = \operatorname{Tr}(a(\lambda)\,\alpha + \beta(\lambda)\,\gamma) + \operatorname{Tr}(c(\lambda)\,\beta + d(\lambda)\,\delta)$. The convergence is uniform for $\lambda \in \gamma$ and it follows that $\lim_n \det(G_n) = \exp \int_\gamma \{\operatorname{Tr}(a(\lambda)\,\alpha + b(\lambda)\gamma) + \operatorname{Tr}(c(\lambda)\,\beta + d(\lambda)\,\delta)\}\, d\lambda$. This finishes the proof that $\det(\cdot)$ extends continuously to $GL_Q(H)$. Q.E.D.

We will now use this result to show that T_0 in (4.6) is surjective. This is equivalent to showing that $GL_{\text{res}}(H) = SL_Q(H) \cdot GL(H_+ \oplus H_-)$. Since Fredholm operators with index 0 are invertible modulo the trace class it is easy to see that $GL_{\text{res}}(H) = GL_Q(H) \cdot GL(H_+ \oplus H_-)$. To prove the desired result it will thus suffice to show that $GL_Q(H) \subseteq SL_Q(H) \cdot GL(H_+ \oplus H_-)$. Suppose $G \in GL_Q(H)$ and let $G_n \in GL_0(H)$ denote a sequence which converges to G in $GL_Q(H)$. Let $\Delta_n = \det(G_n)$ and let e denote a unit vector in H_+. Define a linear map A_n so that $A_n e = \Delta_n e$ and $A_n \equiv I$ on the orthogonal complement of e in H. Then $A_n \in GL(H_+ \oplus H_-)$, $\det A_n = \Delta_n$, and A_n converges in $GL(H_+ \oplus H_-)$ by Lemma 4.3. Thus $G = \lim_n G_n = \lim(G_n A_n^{-1}) \lim_n A_n$ and $\det(G_n A_n^{-1}) = 1$. This shows that $G \in SL_Q(H) \cdot GL(H_+ \oplus H_-)$ and finishes the proof of (4.6).

The representation of $\hat{G}L_Q(H)$ as $SL_Q(H) \times_0 GL(H_+ \oplus H_-)/\ker$ is convenient for understanding this group as a subgroup of $\widehat{\operatorname{Spin}}_Q(W)$ but it is not the simplest realization. Let $GL_d(H)$ denote the subgroup of $GL_{\text{res}}(H)$ consisting of those elements of $GL_{\text{res}}(H)$ which have "d" matrix elements which are trace class perturbations of the identity on H_-. In what follows we identify $GL(H_+)$ and $GL(H_-)$ with subgroups of $GL(H_+ \oplus H_-)$ via the homomorphisms $a \to a \oplus 1$ and $d \to 1 \oplus d$. The map from $SL_Q(H) \times_0 GL(H_+)$ to $GL_Q(H)$ which sends $G \times a \to G \times (a \oplus 1) \ker$ is a homomorphism with kernel $K = \{G \times a \mid G \in SL_Q(H) \text{ and } G = a^{-1} \oplus 1\}$. Lemma 4.3 implies that the map $SL_Q(H) \times_0 GL(H_+) \ni G \times a \to G(a \oplus 1) \in GL_d(H)$ is surjective. The kernel of this homomorphism is also K. Thus $SL_Q(H) \times_0 GL(H_+)/K \simeq GL_d(H)$ and we have a homomorphism from $GL_d(H)$ into $\hat{G}L_Q(H)$. The homomorphism $\Gamma_Q \times \Gamma$ from $\hat{G}L_Q(H)$ into $L(\mathscr{D})$ restricts to a homomorphism from $GL_d(H)$ into $L(\mathscr{D})$ which we denote by Γ_Q to avoid introducing extra notation. Combining this Γ_Q with

Γ we have a homomorphism $\Gamma_Q \times \Gamma$ from $GL_d(H) \times_0 GL(H_-)$ to $L(\mathcal{D})$ given by $\Gamma_Q \times \Gamma(G \times d) = \Gamma_Q(G)\Gamma(I \oplus d)$. We leave to the reader the verification that

$$\hat{G}L_Q(H) \simeq GL_d(H) \times_0 GL(H_-)/\ker(\Gamma_Q \times \Gamma). \tag{4.7}$$

Observe that $\ker(\Gamma_Q \times \Gamma) = \{G \times d \in GL_d(H) \times_0 GL(H_-) \mid G = (I \oplus d)^{-1}$ and $\det(d) = 1\}$. This realization of $\hat{G}L_Q(H)$ makes possible a direct comparison with the group $\hat{G}L(H)$ constructed by G. Segal and G. Wilson (see [35, p. 5]). The groups involved in (4.7) are all "classical groups" and we can summarize the results without direct reference to the spin group. In fact, the calculations in Proposition 4.2 may be used to construct $\hat{G}L_Q(H)$ directly along the lines of Section 3 for $\mathrm{Spin}_Q(W)$ but with considerable simplifications arising from the fact that all the groups which occur in this construction are easily described subgroups of $GL(H)$. We conclude this section with such a summary.

Let $W = H \oplus \bar{H}$ where H is a complex Hilbert space with Hermitian symmetric from $\langle \cdot, \cdot \rangle$ conjugate linear in the second slot. The pairing $(x_1 \oplus y_1, x_2 \oplus y_2) = \langle x_1, y_2 \rangle + \langle x_2, y_1 \rangle$ is then a distinguished bilinear form on W. Let Q_H denote a symmetric involution on H and write $Q = Q_H \oplus (-Q_H)$ for the corresponding involution on W. Write $[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}]$ for the "matrix" of a linear transformation on H relative to the decomposition $H = H_+ \oplus H_-$ determined by the $+1$ and -1 eigenspaces of the involution Q_H. Let $GL_d(H)$ denote the group of invertible linear transformations of H which have "b" and "c" matrix elements in the Schmidt class and a "d" matrix element which is a trace class perturbation of the identity on H_-. Note that the "a" matrix element of such a linear transformation is necessarily a Fredholm map of index 0. Identify $GL(H_-)$ with the linear transformations on H of the form $I \oplus d$ where $d \in GL(H_-)$ and write $GL_d(H) \times_0 GL(H_-)$ for the semi-direct product with the obvious action of $GL(H_-)$ on $GL_d(H)$. Let K denote the normal subgroup of $GL_d(H) \times_0 GL(H_-)$ given by $\{G \times d \in GL_d(H) \times_0 GL(H_-) \mid G = (I \oplus d)^{-1}$ and $\det(d) = 1\}$. Write $\hat{G}L_Q(H) = GL_d(H) \times_0 GL(H_-)/K$. Let T denote the homomorphism from $\hat{G}L_Q(H)$ to the complex orthogonals on W given by $T(G \times d) = [G(I \oplus d)] \oplus [G(I \oplus d)]^{*-1}$. Finally recall that $\mathcal{D}$ is the dense invariant subspace of $A(W_+)$ constructed at the end of Section 3. We have:

THEOREM 4.4. *There exist homomorphisms $\Gamma_Q: GL_d(H) \to L(\mathcal{D})$ and $\Gamma: GL(H_-) \to L(\mathcal{D})$ such that $\Gamma_Q \times \Gamma$ induces a homomorphism $\hat{\Gamma}$ from $\hat{G}L_Q(H)$ into $L(\mathcal{D})$. Furthermore one has*

$$\hat{\Gamma}(g)\, F_Q(x) = F_Q(T(g)x)\, \hat{\Gamma}(g) \qquad \text{for} \quad g \in \hat{G}L_Q(H) \text{ and } x \in W.$$

5. The Borel–Weil Realization of Fock Representations for $\mathrm{Spin}_C(W)$

In this section we give a brief account of the Borel–Weil construction for spin representations. It is convenient to avoid technical issues by concentrating on the finite dimensional situation. Let W denote a finite dimensional complex vector space with non-degenerate bilinear form $(\cdot,\cdot)$ and a distinguished isotropic splitting $W = W_+ \oplus W_-$ associated with the skew symmetric involution Q. Let $G = \mathrm{Spin}_{\mathbb{C}}(W)$ and let $B = \{b \in \mathrm{Spin}_{\mathbb{C}}(W) \mid T(b) = [\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}]\}$. Here $[\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}]$ is the matrix of $T(b)$ relative to the $W_+ \oplus W_-$ splitting of W. Let χ be a homomorphism from B to $\mathbb{C}^*$. Define

$$E = \{(g, \lambda) \in G \times \mathbb{C}^*\}/\sim,$$

where the equivalence classes of the relation $\sim$ are $\{(gb, \chi(b)\lambda) \mid b \in B\}$. Then E may be regarded as a line bundle over G/B with projection $(g, \lambda) \to gB$. The group G acts on E and G/B as follows:

$$\pi_E(g_1)(g, \lambda) = (g_1 g, \lambda)$$

$$\pi_{G/B}(g_1)\, gB = g_1 gB.$$

Since $F_Q(h)^\tau = F_{-Q}(h^\tau)$ (see Section 2) the representation contragradient to $g \to F_Q(g)$ $(g \in G)$ may be identified with $g \to F_{-Q}(g)$. The elements of G/B may be identified with the projective orbit of 1_{-Q} in the contragradient representation as

$$gB = l(F_{-Q}(g)\, 1_{-Q}),$$

where $l(v) =$ line through v. All one needs for this to work is that $F_{-Q}(b)$ preserves the line through 1_{-Q} for $b \in B$. The subgroup B of G was chosen precisely so this would happen. Indeed $F_Q(b)\, 1_{-Q} = \langle b \rangle_{-Q} 1_{-Q}$. The map $B \ni b \to \langle b \rangle_{-Q}$ is a homomorphism (we also know that $\langle b \rangle_{-Q}^2 = \det A(b)$). If we choose $\chi(b) = \langle b \rangle_{-Q}$ then we may identify the fiber in E over gB as the dual of the line $l(F_{-Q}(g)\, 1_{-Q})$. To make this identification define the action of (g, λ) on the vector $\alpha F_{-Q}(g)\, 1_{-Q}$ as

$$(g, \lambda)[\alpha F_{-Q}(g)\, 1_{-Q}] = \lambda\alpha.$$

To see that this does not depend on the choice of g (over gB) consider

$$(gb, \chi(b)\lambda)[\alpha' F_{-Q}(gb)\, 1_{-Q}] = \alpha'\chi(b)\lambda.$$

But $\alpha' F_{-Q}(gb)\, 1_{-Q} = \alpha F_{-Q}(g)\, 1_{-Q}$ so that $\alpha'\chi(b) = \alpha$ or $\alpha'\chi(b)\lambda = \alpha\lambda$.

Next we consider the action of G on sections of E. If σ is a section of E

and $h \in \mathrm{Spin}_{\mathbb{C}}(W)$ then $h\sigma(\cdot) = \pi_E(h)\sigma(\pi_{G/B}(h^{-1})\cdot)$. Let $v \in A(W_+)$ and regard v as an element of the dual of $A(W_-)$. Then v may be regarded as a section of E with

$$v_{gB}[\alpha F_{-Q}(g)\, 1_{-Q}] = (v, \alpha F_{-Q}(g)\, 1_{-Q}) = \alpha(v, F_{-Q}(g)\, 1_{-Q}).$$

If we compare this with $(g, \lambda)[\alpha F_{-Q}(g)\, 1_{-Q}] = \alpha\lambda$ then we see that

$$v_{gB} = (g, (v, F_{-Q}(g)\, 1_{-Q})).$$

We now compare the action $F_Q(h)$ $(h \in G)$ on vectors in $A(W_+)$ with the action of G on sections of E. $[F_Q(h)v]_{hgB}[\alpha F_{-Q}(hg)\, 1_{-Q}] = (F_Q(h)v, \alpha F_{-Q}(hg)\, 1_{-Q}) = (v, \alpha F_{-Q}(h^\tau) F_{-Q}(hg)\, 1_{-Q}) = \alpha(v, F_{-Q}(g)\, 1_{-Q})$ since $h^\tau h = 1$ for $h \in \mathrm{Spin}_{\mathbb{C}}(W)$. Thus $[F_Q(h)v]_{hgB} = (hg, (v, F_{-Q}(g)\, 1_{-Q})) = \pi_E(h)\, v_{gB}$. This shows that the action on sections of E agrees with the Fock representation. It is a consequence of the Borel–Weil theory that G/B is a holomorphic manifold, E is a holomorphic bundle over G/B, and every holomorphic section of E is given by a vector $v \in A(W_+)$ [38].

We now identify the homogeneous space G/B in slightly different terms. The map $\mathrm{Spin}_{\mathbb{C}}(W) \ni g \to T(g)T(B) \in SO_{\mathbb{C}}(W)/T(B)$ is surjective. It is easy to see that g_1 and g_2 go into the same point under this map if and only if $g_1^{-1} g_2 \in B$. Thus one has the isomorphism

$$\mathrm{Spin}_{\mathbb{C}}(W)/B \simeq SO_{\mathbb{C}}(W)/T(B).$$

The homogeneous space $SO_{\mathbb{C}}(W)/T(B)$ may be identified with a (sub) Grassmannian in a standard fashion. Let $G \in SO_{\mathbb{C}}(W)$. Then GW_+ is a maximal isotropic subspace such that $Q_+ : GW_+ \to W_+$ has an even dimensional kernel. Let $Gr_Q(W) = \{V \mid V$ is a maximal isotropic subspace of W and $Q_+ : V \to W_+$ has an even dimensional kernel$\}$. It is not hard to see that $SO_{\mathbb{C}}(W)/T(B) \simeq Gr_Q(W)$. One may prove this using the fact that every maximal isotropic subspace of W has an isotropic complement and that a complex orthogonal with matrix $(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix})$ is in $SO_{\mathbb{C}}(W)$ if and only if A has an even dimensional kernel.

Next consider the situation $W = H \oplus \bar{H}$ with $W_+ = H_+ \oplus \bar{H}_-$. We are interested in the subgroup of $\mathrm{Spin}_{\mathbb{C}}(W)$ with induced rotations of the form $A \oplus A^{*-1}$ where $A \in GL(H)$. Because these all have determinant 1 (remember $\det(A^* \mid \bar{H}) = \det(A \mid H)$) they are in $SO_{\mathbb{C}}(W)$ automatically. The action of $A \oplus A^{*-1}$ on W_+ is $(A \mid H_+) \oplus (A^{*-1} \mid \bar{H}_-)$. Thus we may identify the appropriate orbit space as $GL(H)\, H_+$ (the Grassmannian of subspaces with half the dimension of H). We want to identify the restricted bundle $E \to GL(H) \cdot H_+$ with the $\det^*$ bundle over $GL(H) H_+$. An easy way to see how to do this is to recall that in finite dimensions all Fock representations are unitarily equivalent. Thus there exists a unitary $U: A(W_-) \to A(H)$ so that $UF_{-Q}(g) = F_0(g)U$ where $F_0(\cdot)$ is the Q_0 Fock representation ($Q_0 = [\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}]$ on $H \oplus \bar{H}$). Thus the line through $F_{-Q}(g)\, 1_{-Q}$ may be identified with the line through $F_0(g)\, U1_{-Q}$. However, the vector $F_0(g)\, U1_{-Q}$ is determined up to a multiple by the fact that it is annihilated by $F_0(T(g)w)$ for all $w \in W_+$. Let $T(g) = A \oplus A^{*-1}$. Then $F_0(T(g)w) = a^*(Aw_+) + a(A^{-\tau}w_-)$ where $w_\pm \in H_\pm$. Let $w_1, ..., w_n$ be a basis for H_+. Then $Aw_1 \wedge Aw_2 \cdots \wedge Aw_n$ is annihilated by $a^*(Aw_+)$ and by $a(A^{-\tau}, w_-)$ for $w_\pm \in H_\pm$. Thus the fibers of E over $GL(H)H_+$ are dual to the lines through $Aw_1 \wedge \cdots \wedge Aw_n$. Thus $E \simeq \det^*(GL(H)\, H_+)$.

Segal and Wilson [35] have shown how to extend the definition of the det* bundle to the infinite dimensional setting (in the algebraic setting this was done earlier by Kac and Peterson [10]). We leave the reader the interesting problem of making a similar geometric construction of the bundle E over $Gr_Q(W)$.

Note added in proof. Since this paper was written, G. Segal and A. Pressley have published an elegant construction of the infinite complex spin groups in Chapter 5 of their book on Loop Groups [39]. Their construction avoids the consideration of Clifford algebras altogether!

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